Vector and Tensor Calculus
An Introduction

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## Contents

1 Mathematical Prerequisites 1  
1.1 Basics of vector calculus 1

2 Fundamentals of tensor calculus 9  
2.1 Introduction of the tensor concept 9  
2.2 Basic rules of tensor algebra 10  
2.3 Specific tensors and operations 13  
2.4 Change of the basis 16  
2.5 Higher order tensors 25  
2.6 Fundamental tensor of 3rd order (RICCI permutation tensor) 28  
2.7 The axial vector 28  
2.8 The outer tensor product of tensors 33  
2.9 The eigenvalue problem and the invariants of tensors 34

3 Fundamentals of vector and tensor analysis 36  
3.1 Introduction of functions 36  
3.2 Functions of scalar variables 36  
3.3 Functions of vector and tensor variables 37  
3.4 Integral theorems 42  
3.5 Transformations between actual and reference configurations 47
1 Mathematical Prerequisites

1.1 Basics of vector calculus

(a) Symbols, summation convention, Kronecker $\delta$

Single- or multiple subscripts

\[
\begin{align*}
  u_i & \rightarrow u_1, u_2, u_3, \ldots \\
  u_i v_k & \rightarrow u_1 v_1, u_1 v_2, u_1 v_3, \ldots \\
  & \quad u_2 v_1, u_2 v_2, \ldots \\
  \vdots & \\
  t_{ik} & \rightarrow t_{11}, t_{12}, \ldots \\
  \vdots & 
\end{align*}
\]

Summation convention of Einstein

**Definition:** Whenever the same subscript occurs twice in a term, a summation over that “double” subscript has to be carried out.

Example:

\[
\begin{align*}
  u_j v_j &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n, \\
  &= \sum_{j=1}^{n} u_j v_j
\end{align*}
\]

Kronecker symbol

**Definition:** It exists a symbol $\delta_{ik}$ with the following properties

\[
\delta_{ik} = \begin{cases} 
0 & \text{if } i \neq k \\
1 & \text{if } i = k 
\end{cases}
\]

Example:

\[
\begin{align*}
  u_i \delta_{ik} &= u_1 \delta_{1k} + u_2 \delta_{2k} + \ldots + u_n \delta_{nk} \\
  \text{with } u_1 \delta_{1k} &= \begin{cases} 
  u_1 \delta_{11} &= u_1 \\
  u_1 \delta_{12} &= 0 \\
  \vdots & \\
  u_1 \delta_{1n} &= 0 
\end{cases} \\
  \rightarrow u_i \delta_{ik} &= u_k
\end{align*}
\]

If the Kronecker symbol is multiplied with another quantity and if there is a double subscript in this term, the Kronecker symbol disappears, the “double” subscript can be dropped and the free subscript remains.
Rem.: Subscripts occurring two times in a term can be renamed arbitrarily.

(b) TERMS AND DEFINITIONS OF VECTOR ALGEBRA

Rem.: The following statements are related to the standard three-dimensional (3-d) physical space, i.e. the Euclidean vector space $\mathcal{V}^3$.
Generally, SPACE is a mathematical concept of a set and does not directly refer to the 3-d point space $\mathcal{E}^3$ and the 3-d vector space $\mathcal{V}^3$.

A: Vector addition

Requirement: $\{u, v, w, \ldots\} \in \mathcal{V}^3$

The following relations hold:

\[
\begin{align*}
    u + v &= v + u : \text{commutative law} \\
    u + (v + w) &= (u + v) + w : \text{associative law} \\
    u + 0 &= u : 0 : \text{identity element of vector addition} \\
    u + (-u) &= 0 : -u : \text{inverse element of vector addition}
\end{align*}
\]

Examples to the commutative and the associative law:

B: Multiplication of a vector with a scalar quantity

Requirement: $\{u, v, w, \ldots\} \in \mathcal{V}^3$; $\{\alpha, \beta, \ldots\} \in \mathbb{R}$

\[
\begin{align*}
    1 \cdot v &= v : 1: \text{identity element} \\
    \alpha (\beta v) &= (\alpha \beta) v : \text{associative law} \\
    (\alpha + \beta) v &= \alpha v + \beta v : \text{distributive law (addition of scalars)} \\
    \alpha (v + w) &= \alpha v + \alpha w : \text{distributive law (addition of vectors)} \\
    \alpha v &= v \alpha : \text{commutative law}
\end{align*}
\]

Rem.: In the general vector calculus, the definitions A and B constitute the "affine vector space".

Linear dependency of vectors

Rem.: In $\mathcal{V}^3$, 3 non-coplanar vectors are linearly independent; i.e. each further vector can be expressed as an multiple of these vectors.

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**Theorem:** The vectors \( \mathbf{v}_i \ (i = 1, 2, 3, \ldots, n) \) are linearly dependent, if real numbers \( \alpha_i \) exist which are not all equal to zero, such that

\[
\alpha_i \mathbf{v}_i = \mathbf{0} \quad \text{or} \quad \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}
\]

Example (plane case):

\[
\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \neq \mathbf{0}
\]

but:

\[
\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0}
\]

\( \rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \): linearly dependent

\( \rightarrow \{\mathbf{v}_1, \mathbf{v}_2\} \): linearly independent

**Rem.:** The \( \alpha_i \) can be multiplied by any factor \( \lambda \).

**Basis vectors in \( \mathbb{V}^3 \)**

ex.: \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) : linearly independent

then: \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}\} \) : linearly dependent

Thus, it follows that

\[
\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \lambda \mathbf{v} = \mathbf{0}
\]

\( \rightarrow \lambda \mathbf{v} = -\alpha_i \mathbf{v}_i \)

or \( \mathbf{v} = -\frac{\alpha_i}{\lambda} \mathbf{v}_i =: \beta_i \mathbf{v}_i \)

with \( \{ \beta_i = -\frac{\alpha_i}{\lambda} \ : \ \text{coefficients (of the vector components)} \}

\( \mathbf{v}_i \ : \ \text{basis vectors of } \mathbf{v} \)

**Choice of a specific basis**

**Rem.:** In \( \mathbb{V}^3 \), each system of 3 linearly independent vectors can be selected as a basis; e. g.

\( \mathbf{v}_i \) : general basis

\( \mathbf{e}_i \) : specific, orthonormal basis (Cartesian, right-handed)
Representation of the vector $v$:

$$v = \begin{cases} \beta_i v_i \\ \gamma_i e_i \end{cases}$$

**here:** Specific choice of the Cartesian basis system $e_i$

**Notations**

$$v = v_i e_i = v_1 e_1 + v_2 e_2 + v_3 e_3$$

with

$$\begin{cases} 
    v_i e_i : \text{vector components} \\
    v_i : \text{coefficients of the vector components}
\end{cases}$$

**C: Scalar product of vectors**

The following relations hold:

$$u \cdot v = v \cdot u$$ : commutative law

$$u \cdot (v + w) = u \cdot v + u \cdot w$$ : distributive law

$$\alpha (u \cdot v) = u \cdot (\alpha v) = (\alpha u) \cdot v$$ : associative law

$$u \cdot v = 0 \quad \forall u, \text{ if } v \equiv 0$$

$$\rightarrow u \cdot u \neq 0 \quad , \text{ if } u \neq 0$$

**Rem.:** The definitions A, B and C constitute the “Euclidean vector space”. If instead of $u \cdot u \neq 0$ especially

$$u \cdot u > 0 \quad , \text{ if } u \neq 0,$$

holds, then A, B and C define the “proper Euclidean vector space $\mathbb{V}^3$” (physical space).

**Square and norm of a vector**

$$v^2 := v \cdot v \quad , \quad v = |v| = \sqrt{v^2}$$

**Rem.:** The norm is the value or the positive square root of the vector.

**Angle between two vectors**

$$\dot{\Phi} (u; v) =: \alpha$$
Law of cosines

\[ |u - v|^2 = |u|^2 + |v|^2 - 2 |u| |v| \cos \alpha \]

\[ \rightarrow \quad \cos \alpha = \frac{u^2 + v^2 - (u - v)^2}{2 |u| |v|} \]

or

\[ u \cdot v = |u| |v| \cos \alpha \]

Scalar products (inner products) in an orthonormal basis

Scalar product of the basis vectors \( e_i \):

\[
\langle e_i \mid e_k \rangle \begin{cases} 
90^\circ & \text{if } i \neq k : \cos 90^\circ = 0 \\
0^\circ & \text{if } i = k : \cos 0^\circ = 1 
\end{cases}
\]

thus

\[ e_i \cdot e_k = |e_i| |e_k| \cos \langle e_i \mid e_k \rangle = \cos \langle e_i \mid e_k \rangle \]

It follows with the Kronecker \( \delta \)

\[ e_i \cdot e_k = \delta_{ik} = \begin{cases} 
1 & \text{if } i = k \\
0 & \text{if } i \neq k 
\end{cases} \]

Scalar product of two vectors:

\[ u \cdot v = (u_i e_i) \cdot (v_k e_k) = u_i v_k (e_i \cdot e_k) = u_i v_k \delta_{ik} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \]

D: Vector or cross product (outer product) of vectors

One defines the following vector product

\[ u \times v = |u| |v| \sin \langle u \mid v \rangle n \]

with \( n \): unit vector \( \perp u, v \) (corkscrew rule or right-hand rule, see page 7)

From the above definition, the following relations can be derived

\[ u \times v = -v \times u \quad : \text{no commutative law} \]

\[ u \times (v + w) = u \times v + u \times w \quad : \text{distributive law} \]

\[ \alpha (u \times v) = (\alpha u) \times v = u \times (\alpha v) \quad : \text{associative law} \]
Scalar triple product (parallelepidal product):
\[ u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v) \]

Arithmetic laws for the vector product (without proof)
\[ u \times u = 0 \]
\[ (u + v) \times w = u \times w + v \times w \]
\[ u \cdot (u \times v) = v \cdot (u \times u) = 0 \]

Expansion theorem:
\[ u \times (v \times w) = (u \cdot w) v - (u \cdot v) w \]

LAGRANGEan identity (Jean Louis Lagrange: 1736-1813):
\[ (u \times v) \cdot (w \times z) = (u \cdot w) (v \cdot z) - (u \cdot z) (v \cdot w) \]

Norm of the vector product:
\[ |u \times v| = |u||v| \sin \gamma(u; v) \]

Vector product in an orthonormal basis
here: simplified representation in matrix notation
Calculation of
\[ u = v \times w = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2) e_1 - (v_1 w_3 - v_3 w_1) e_2 + (v_1 w_2 - v_2 w_1) e_3 \]

Rem.: \( u \perp v, w \); i. e. \( u \cdot v = u \cdot w = 0 \) holds

Example:
\[ u \cdot v = u_i v_i = (v_2 w_3 - v_3 w_2) v_1 - (v_1 w_3 - v_3 w_1) v_2 + (v_1 w_2 - v_2 w_1) v_3 = 0 \quad \text{q. e. d.} \]

Remarks on the products between vectors
\bullet on the scalar product

Decomposition of a vector (example: in 2-d):
\[ u = u_1 + u_2 \]
with \( u_1 = u_1 e_1 \) and \( u_2 = u_2 e_2 \)
\( u_1, u_2 \): vector components
\( u_1, u_2 \): coefficients of the vector components
Projection of \( \mathbf{u} \) on the directions of \( \mathbf{e}_i \):

\[
\mathbf{u}_i = \mathbf{u} \cdot \mathbf{e}_i
\]

Verification of the projection law:

\[
\mathbf{u} \cdot \mathbf{e}_i = (u_k \mathbf{e}_k) \cdot \mathbf{e}_i = u_k \delta_{ki} = u_i \quad \text{q. e. d.}
\]

Calculation of the projections:

\[
\begin{align*}
    u_1 &= |\mathbf{u}| |\mathbf{e}_1| \cos \alpha \\
         &= |\mathbf{u}| \cos \alpha = u \cos \alpha \\
    u_2 &= u \cos \beta \\
         &= u \cos (90^\circ - \alpha) = u \sin \alpha
\end{align*}
\]

\[\textbf{Note:}\] For the values of the vector components, the following relations hold

\[
\begin{align*}
    u_1 &= u \cos \alpha \\
    u_2 &= u \sin \alpha
\end{align*}
\]

• on the \textit{vector product}

Orientation of the vector \( \mathbf{u} = \mathbf{v} \times \mathbf{w} \):

It is obvious that

\[
\begin{align*}
    \mathbf{z} &= \mathbf{w} \times \mathbf{v} \\
    \mathbf{v} \times \mathbf{w} &= -\mathbf{w} \times \mathbf{v}
\end{align*}
\]

Value of the vector product:

\[
|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}| |\mathbf{w}| \sin \alpha = v (w \sin \alpha)
\]
Note: The vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ (corkscrew orientation); its value corresponds to the area spanned by $\mathbf{v}$ and $\mathbf{w}$.

Scalar triple product (parallelepiedal product):

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =: [\mathbf{u} \mathbf{v} \mathbf{w}]$$

with $\mathbf{z} = \mathbf{v} \times \mathbf{w}$

follows $\mathbf{u} \cdot \mathbf{z} = |\mathbf{u}| |\mathbf{z}| \cos \gamma$

$$= z (u \cos \gamma)$$

with $(u \cos \gamma)$: projection of $\mathbf{u}$ on the direction of $\mathbf{z}$

Rem.: The parallelepiedal product yields the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Remark: The preceding and the following relations are valid with respect to an arbitrary basis system. For simplicity, the following material is restricted to the orthonormal basis, whenever a basis notation occurs. Concerning a more general basis representation, cf., e. g., DE BOER, R.: Vektor- und Tensorrechnung für Ingenieure. Springer-Verlag, Berlin 1982.
2 Fundamentals of tensor calculus

Rem.: The following statements are related to the proper Euklidian vector space \( \mathcal{V}^3 \) and the corresponding dyadic product space \( \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \) (\( n \)-times) of \( n \)-th order.

2.1 Introduction of the tensor concept

(a) Tensor concept and linear mapping

**Definition:** A 2nd order (2nd rank) tensor \( T \) is a linear mapping which transforms a vector \( u \) uniquely in a vector \( w \):

\[
w = T u
\]

therein:

\[
\begin{align*}
\{ u, w \} & \in \mathcal{V}^3 ; \quad T \in \mathcal{L}(\mathcal{V}^3, \mathcal{V}^3) \\
\mathcal{L}(\mathcal{V}^3, \mathcal{V}^3) & : \text{set of all 2nd order tensors or linear mappings of vectors, respectively}
\end{align*}
\]

(b) Tensor concept and dyadic product space

**Definition:** There is a “simple tensor” \((a \otimes b)\) with the property

\[
(a \otimes b) c := (b \cdot c) a
\]

therein:

\[
\begin{align*}
\{ a \otimes b \} & \in \mathcal{V}^3 \otimes \mathcal{V}^3 \quad \text{(dyadic product space)} \\
\otimes & : \text{dyadic product (binary operator of } \mathcal{V}^3 \otimes \mathcal{V}^3) \\
\end{align*}
\]

It follows directly that

\[
a \otimes b \in \mathcal{L}(\mathcal{V}^3, \mathcal{V}^3) \quad \rightarrow \quad \mathcal{V}^3 \otimes \mathcal{V}^3 \subset \mathcal{L}(\mathcal{V}^3, \mathcal{V}^3)
\]

Rem.: \((a \otimes b)\) maps a vector \( c \) onto a vector \( d = (b \cdot c) a \).

Basis notation of a simple tensor:

\[
A := a \otimes b = (a_i e_i) \otimes (b_k e_k) = a_i b_k (e_i \otimes e_k)
\]

with \[
\begin{align*}
\{ a_i b_k \} & : \text{coefficients of the tensor components} \\
\{ e_i \otimes e_k \} & : \text{tensor basis}
\end{align*}
\]

Tensors \( A \in \mathcal{V}^3 \otimes \mathcal{V}^3 \) have 9 independent components (and directions); e. g. \( a_1 b_3 (e_1 \otimes e_3) \) etc.
Introduction of arbitrary tensors $T \in V^3 \otimes V^3$:

$$T = t_{ik} (e_i \otimes e_k)$$

with $t_{ik} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$:

- matrix of coefficients of $T$ with 9 independent quantities

### 2.2 Basic rules of tensor algebra

Requirement: $\{A, B, C, \ldots\} \in V^3 \otimes V^3$.

#### (a) Tensor Addition

- $A + B = B + A$ : commutative law
- $A + (B + C) = (A + B) + C$ : associative law
- $A + 0 = A$ : 0 : identical element
- $A + (-A) = 0$ : $-A$ : inverse element

Tensor addition with respect to an orthonormal tensor basis:

$$A = a_{ik} (e_i \otimes e_k), \quad B = b_{ik} (e_i \otimes e_k)$$

$$\rightarrow C = A + B = \left(\sum c_{ik} \right)(e_i \otimes e_k)$$

**Rem.:** A tensor addition carried out as an addition of the tensor coefficients requires that both tensors have the same tensor basis.

#### (b) Multiplication of Tensors by a Scalar

- $1 \cdot A = A$ : 1 : identical element
- $\alpha \left( \beta A \right) = (\alpha \beta) A$ : associative law
- $(\alpha + \beta) A = \alpha A + \beta A$ : distributive law (with respect to the addition of scalars)
- $\alpha \left( A + B \right) = \alpha A + \alpha B$ : distributive law (with respect to the addition of tensors)
- $\alpha A = A \alpha$ : commutative law

#### (c) Linear Mapping between Tensor and Vector

The following definitions make use of the linear mapping (cf. 2.1)

$$w = T u$$
Rem.: In the literature, the multiplication of a vector by a tensor is also called “contraction”.

The following relations hold:

\[ A(u + v) = Au + Av \] : distributive law

\[ A(\alpha u) = \alpha(Au) \] : associative law

\[ (A + B)u = Au + Bu \] : distributive law

\[ (\alpha A)u = \alpha (Au) \] : associative law

\[ 0u = 0 \] : \(0\) : zero element of the linear mapping

\[ Iu = u \] : \(I\) : identity element of the linear mapping

Linear mapping in basis notation:

\[ A = a_{ik} (e_i \otimes e_k) , \quad u = u_i e_i \]

One obtains

\[ w = Au = a_{ik} u_j \delta_{kj} e_i = \underbrace{a_{ik} u_k}_{w_i} e_i \] mit \(\{\begin{array}{l}
    i : \text{free index (basis index)} \\
    k : \text{silent index (double index of } w_i) 
\end{array}\) 

Rem.: In general, a linear mapping \(A\) causes both a rotation and a stretch of a vector \(u\).

Identity tensor \(I \in \mathcal{V}^3 \otimes \mathcal{V}^3\):

\[ I = \delta_{ik} e_i \otimes e_k = e_i \otimes e_i \]

Proof of the defining property:

\[ u = Iu = (e_i \otimes e_i) u_j e_j = u_j (e_i \otimes e_i) e_j = u_j \delta_{ij} e_i = u_i e_i \quad \text{q. e. d.} \]

Rem.: Tensors built from basis vectors are called fundamental tensors, i.e.

\[ I \in \mathcal{V}^3 \otimes \mathcal{V}^3 \text{ is the fundamental tensor of 2nd order.} \]

(d) **Scalar Product of Tensors** (inner product)

The following relations hold:

\[ A \cdot B = B \cdot A \] : commutative law

\[ A \cdot (B + C) = A \cdot B + A \cdot C \] : distributive law

\[ (\alpha A) \cdot B = A \cdot (\alpha B) = \alpha (A \cdot B) \] : associative law

\[ A \cdot B = 0 \quad \forall A, \text{ if } B \equiv 0 \]

\[ \rightarrow A \cdot A > 0 \text{ for } A \neq 0 \]
Scalar product of $A$ with a simple tensor $a \otimes b \in \mathcal{V}^3 \otimes \mathcal{V}^3$:

$$A \cdot (a \otimes b) = a \cdot A b$$

Scalar product of $A$ and $B$ in basis notation:

$$A = a_{ik}(e_i \otimes e_k), \quad B = b_{ik}(e_i \otimes e_k)$$

$$\alpha = A \cdot B = a_{ik}(e_i \otimes e_k) \cdot b_{st}(e_s \otimes e_t) = a_{ik} b_{st}(e_i \otimes e_k) \cdot (e_s \otimes e_t)$$

One obtains

$$\alpha = a_{ik} b_{st} \delta_{is} \delta_{kt} = a_{ik} b_{ik}$$

Rem.: The result of the scalar product is a scalar.

### (e) TENSOR PRODUCT OF TENSORS

**Definition:** The tensor product of tensors yields

$$(A B) v = A (B v)$$

**Rem.:** With this definition, the tensor product of tensors is directly linked to the linear mapping (cf. 2.1 (a)).

The following relations hold:

$$(A B) C = A (B C) : \text{associative law}$$

$$A (B + C) = A B + A C : \text{distributive law}$$

$$(A + B) C = A C + B C : \text{distributive law}$$

$$\alpha (A B) = (\alpha A) B = A (\alpha B) : \text{associative law}$$

$$I T = T I = T : I : \text{identity element}$$

$$0 T = T 0 = 0 : 0 : \text{zero element}$$

**Rem.:** In general, the commutative law is not valid, i.e. $A B \neq B A$.

**Tensor product of simple tensors:**

$$A = a \otimes b, \quad B = c \otimes d$$

It follows with the above definition

$$(A B) v = A (B v) \quad \rightarrow \quad [(a \otimes b) (c \otimes d)] v = (a \otimes b) [(c \otimes d) v] = (a \otimes b) (d \cdot v) c = (b \cdot c) (d \cdot v) a = [(b \cdot c) (a \otimes d)] v$$

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Consequence:

\[(a \otimes b) (c \otimes d) = (b \cdot c) a \otimes d\]

Tensor product in basis notation:

\[
A \otimes B = a_{ik}(e_i \otimes e_k) b_{st}(e_s \otimes e_t)
\]

\[
= a_{ik} b_{st} (e_i \otimes e_k) (e_s \otimes e_t)
\]

\[
= a_{ik} b_{st} \delta_{ks} (e_i \otimes e_t)
\]

\[
= a_{ik} b_{kt} (e_i \otimes e_t)
\]

Rem.: The result of a tensor product is a tensor.

2.3 Specific tensors and operations

(a) Transposed tensor

**Definition:** The transposed tensor \(A^T\) belonging to \(A\) exhibits the property

\[w \cdot (A u) = (A^T w) \cdot u\]

The following relations hold:

\[(A + B)^T = A^T + B^T\]

\[(\alpha A)^T = \alpha A^T\]

\[(A B)^T = B^T A^T\]

Transposition of a simple tensor \(a \otimes b\):

It follows with the above definition

\[w \cdot (a \otimes b) u = w \cdot (b \cdot u) a\]

\[= (w \cdot a) (b \cdot u)\]

\[= (b \otimes a) w \cdot u\]

\[\rightarrow (a \otimes b)^T = b \otimes a\]

Transposed tensor in basis notation:

\[A = a_{ik}(e_i \otimes e_k)\]

\[\rightarrow A^T = a_{ik}(e_k \otimes e_i)\]

\[= a_{ki}(e_i \otimes e_k) : \text{renaming the indices}\]
(b) **SYMMETRIC AND SKEW-SYMMETRIC TENSOR**

**Definition:** A tensor \( A \in \mathcal{V}^3 \otimes \mathcal{V}^3 \) is symmetric, if

\[
A = A^T
\]

and skew-symmetric (antimetric), if

\[
A = -A^T
\]

Symmetric and skew-symmetric parts of an arbitrary tensor \( A \in \mathcal{V}^3 \otimes \mathcal{V}^3 \):

\[
sym A = \frac{1}{2} (A + A^T)
\]

\[
skw A = \frac{1}{2} (A - A^T)
\]

\[\rightarrow A = sym A + skw A\]

Properties of symmetric and skew-symmetric tensors:

\[
w \cdot (sym A) v = (sym A) w \cdot v
\]

\[
v \cdot (skw A) v = -(skw A) v \cdot v = 0
\]

Positive definite symmetric tensors:

- \( sym A \) is positive definite, if \( sym A \cdot (v \otimes v) = v \cdot (sym A) v > 0 \)
- \( sym A \) is positive semi-definite, if \( sym A \cdot (v \otimes v) = v \cdot (sym A) v \geq 0 \)

(c) **INVERSE TENSOR**

**Definition:** If \( A^{-1} \) inverse to \( A \) exists, it exhibits the property

\[
v = A w \quad \leftrightarrow \quad w = A^{-1} v
\]

The following relations hold:

\[
AA^{-1} = A^{-1}A = I
\]

\[
(A^{-1})^T = (A^T)^{-1} =; A^{T-1}
\]

\[
(AB)^{-1} = B^{-1}A^{-1}
\]
The computation of the inverse tensor in basis notation is carried out by introducing the “double cross product” (outer tensor product of tensors), cf. 2.8.

**Orthogonal Tensor**

**Definition:** An orthogonal tensor $Q \in \mathcal{V}^3 \otimes \mathcal{V}^3$ exhibits the property

\[
Q^{-1} = Q^T \quad \iff \quad Q Q^T = I
\]

Additionally

\[
\begin{cases}
\{ (\det Q)^2 = 1 : \text{orthogonality} \\
\det Q = 1 : \text{proper orthogonality}
\end{cases}
\]

The computation of the determinant of 2nd order tensors is defined with the aid of the double cross product, cf. 2.8.

**Properties of orthogonal tensors:**

\[
Q v \cdot Q w = Q^T Q v \cdot w = v \cdot w
\]

\[
\rightarrow Q u \cdot Q u = u \cdot u
\]

**Illustration:**

- **in general:** linear mapping with $A \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes a rotation and a stretch
- **in special:** linear mapping with $Q \in \mathcal{V}^3 \otimes \mathcal{V}^3$ causes only a rotation

**Trace of a Tensor**

**Definition:** The trace $\text{tr} A$ of a tensor $A \in \mathcal{V}^3 \otimes \mathcal{V}^3$ is the scalar product

\[
\text{tr} A = A \cdot I
\]

The following relations hold:

\[
\begin{align*}
\text{tr} (\alpha A) &= \alpha \text{tr} A \\
\text{tr} (a \otimes b) &= a \cdot b \\
\text{tr} A^T &= \text{tr} A \\
\text{tr} (A B) &= \text{tr} (B A) \\
\rightarrow (A B) \cdot I &= B \cdot A^T = B^T \cdot A \\
\text{tr} (A B C) &= \text{tr} (B C A) = \text{tr} (C A B)
\end{align*}
\]
2.4 Change of the basis

Rem.: The goal is to find a relation between vectors and tensors which belong to different basis systems.

here: Restriction to orthonormal basis systems which are rotated against each other.

(A) Rotation of the basis system

Illustration:

Development of the transformation tensor:

The following relations hold:

\[ \mathbf{\dot{e}}_i = \mathbf{I} \mathbf{\dot{e}}_i \quad \text{and} \quad \mathbf{I} = \mathbf{e}_j \otimes \mathbf{e}_j \]

Thus,

\[ \mathbf{\dot{e}}_i = (\mathbf{e}_j \otimes \mathbf{e}_j) \mathbf{\dot{e}}_i = (\mathbf{e}_j \cdot \mathbf{\dot{e}}_i) \mathbf{e}_j \]

using \( \mathbf{\dot{e}}_i = \delta_{ik} \mathbf{e}_k \) leads to

\[ \mathbf{\dot{e}}_i = (\mathbf{e}_j \cdot \mathbf{\delta}_{ik} \mathbf{\dot{e}}_k) \mathbf{e}_j = (\mathbf{e}_j \cdot \mathbf{\dot{e}}_i) (\mathbf{e}_i \cdot \mathbf{e}_k) \mathbf{e}_j \]

one obtains

\[ \mathbf{\dot{e}}_i = (\mathbf{e}_j \cdot \mathbf{\dot{e}}_k) (\mathbf{e}_j \otimes \mathbf{e}_k) \mathbf{e}_i =: \mathbf{R} \mathbf{e}_i \quad \text{with} \quad \mathbf{R} = (\mathbf{e}_j \cdot \mathbf{\dot{e}}_k) \mathbf{e}_j \otimes \mathbf{e}_k \]

Rem.: \( \mathbf{R} \) is the transformation tensor which transforms the basis vectors \( \mathbf{e}_i \) into the basis vectors \( \mathbf{\dot{e}}_i \).

Coefficient matrix \( R_{jk} \):

\[ R_{jk} = \mathbf{e}_j \cdot \mathbf{\dot{e}}_k = |\mathbf{e}_j| |\mathbf{\dot{e}}_k| \cos \angle (\mathbf{e}_j; \mathbf{\dot{e}}_k) = \cos \alpha_{jk} \quad \text{with} \quad |\mathbf{e}_j| = |\mathbf{\dot{e}}_k| = 1 \]

Rem.: \( R_{jk} \) contains the 9 cosines of the angles between the directions of the basis vectors \( \mathbf{e}_j \) and \( \mathbf{\dot{e}}_k \).
Orthogonality of the transformation tensor:

**Rem.** By $\mathbf{R}$, the basis vectors $\mathbf{e}_i$ are only rotated towards $\mathbf{e}_i^*$, thus, $\mathbf{R}$ is an orthogonal tensor.

Orthogonality condition:

$$\mathbf{R} \mathbf{R}^T = \mathbf{I} = R_{jk} (\mathbf{e}_j \otimes \mathbf{e}_k) R_{pn} (\mathbf{e}_n \otimes \mathbf{e}_p) = R_{jk} R_{pn} \delta_{kn} \mathbf{e}_j \otimes \mathbf{e}_p$$

$$= R_{jk} R_{pk} (\mathbf{e}_j \otimes \mathbf{e}_p)$$

It follows with $\mathbf{I} = \delta_{jp} (\mathbf{e}_j \otimes \mathbf{e}_p)$ by comparison of coefficients

$$R_{jk} R_{pk} = \delta_{jp} \quad (*)$$

**Rem.** $(*)$ contains 6 constraints for the 9 cosines ($\mathbf{R} \mathbf{R}^T = \text{sym} (\mathbf{R} \mathbf{R}^T)$), i.e. only 3 of 9 trigonometrical functions are independent. Thus, the rotation of the basis system is defined by 3 angles.

**(B) Introduction of “CARDANO angles”**

**Idea:** Rotation around 3 axes which are given by the basis directions $\mathbf{e}_i$. This procedure was firstly investigated by GIROLAMO CARDANO (1501-1576).

**Procedure:** The rotation of the basis system is carried out by 3 independent rotations around the axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Each rotation is expressed by a transformation tensor $\mathbf{R}_i$ ($i = 1, 2, 3$).

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$:

$$\mathbf{e}_i = \{ \mathbf{R}_1 [ \mathbf{R}_2 (\mathbf{R}_3 \mathbf{e}_i)] \} = \mathbf{R}_i \mathbf{e}_i \quad \text{with} \quad \mathbf{R} = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3$$

Rotation of $\mathbf{e}_i$ around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{e}_i = \{ \mathbf{R}_3 [ \mathbf{R}_2 (\mathbf{R}_1 \mathbf{e}_i)] \} = \tilde{\mathbf{R}} \mathbf{e}_i \quad \text{with} \quad \tilde{\mathbf{R}} = \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1$$

Obviously,

$$\mathbf{R} \neq \tilde{\mathbf{R}} \quad \rightarrow \quad \mathbf{e} \neq \tilde{\mathbf{e}}_i$$

**Rem.** The result of the orthogonal transformation depends on the sequence of the rotations.
Illustration:
(a) Rotation around $\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1$ (e. g. each about $90^\circ$)

(b) Rotation around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (e. g. each about $90^\circ$)

with

Definition of the orthogonal rotation tensors $\mathbf{R}_i$

(a) Rotation around the $\mathbf{e}_3$-axis

The following relations hold:

\[
\begin{align*}
\dot{\mathbf{e}}_1 &= \cos \varphi_3 \mathbf{e}_1 + \sin \varphi_3 \mathbf{e}_2 \\
\dot{\mathbf{e}}_2 &= -\sin \varphi_3 \mathbf{e}_1 + \cos \varphi_3 \mathbf{e}_2 \\
\dot{\mathbf{e}}_3 &= \mathbf{e}_3
\end{align*}
\]
In general,
\[ \mathbf{e}_i = R_3 \mathbf{e}_i = R_{3jk} (\mathbf{e}_j \otimes \mathbf{e}_k) \mathbf{e}_i = R_{3jk} \delta_{ki} \mathbf{e}_j = R_{3ji} \mathbf{e}_j \]

Thus, by comparison of coefficients
\[ R_3 = R_{3ji} (\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{3ji} = \begin{bmatrix} \cos \varphi_3 & -\sin \varphi_3 & 0 \\ \sin \varphi_3 & \cos \varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(b) Rotation around the \( \mathbf{e}_2 \)- and \( \mathbf{e}_1 \)-axis

Analogously,
\[ R_2 = R_{2ji} (\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{2ji} = \begin{bmatrix} \cos \varphi_2 & 0 & \sin \varphi_2 \\ 0 & 1 & 0 \\ -\sin \varphi_2 & 0 & \cos \varphi_2 \end{bmatrix} \]
\[ R_1 = R_{1ji} (\mathbf{e}_j \otimes \mathbf{e}_i) \quad \text{with} \quad R_{1ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi_1 & -\sin \varphi_1 \\ 0 & \sin \varphi_1 & \cos \varphi_1 \end{bmatrix} \]

Rem. : The rotation tensor \( R \) can be composed of single rotations under consideration of the rotation sequence.

(c) Definition of the total rotation \( R \)

\((c_1)\) it follows from rotation of \( \mathbf{e}_i \) around \( \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1 \) that
\[ \mathbf{R} \rightarrow \mathbf{R} = R_1 R_2 R_3 \]
\[ = R_{1ij} (\mathbf{e}_i \otimes \mathbf{e}_j) R_{2no} (\mathbf{e}_n \otimes \mathbf{e}_o) R_{3pq} (\mathbf{e}_p \otimes \mathbf{e}_q) \]
\[ = R_{1ij} R_{2no} R_{3pq} \delta_{jn} \delta_{op} (\mathbf{e}_i \otimes \mathbf{e}_q) \]
\[ = \underbrace{R_{1ij} R_{2jo} R_{3oj}}_{\mathbf{R}_{iq}} (\mathbf{e}_i \otimes \mathbf{e}_q) \]

with
\[ \mathbf{R}_{iq} = \begin{bmatrix} \cos \varphi_2 \cos \varphi_3 & -\cos \varphi_2 \sin \varphi_3 & \sin \varphi_2 \\ \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 + \cos \varphi_1 \sin \varphi_3 & -\sin \varphi_1 \sin \varphi_2 \sin \varphi_3 + \cos \varphi_1 \cos \varphi_3 & -\sin \varphi_1 \cos \varphi_2 \\ -\cos \varphi_1 \sin \varphi_2 \cos \varphi_3 + \sin \varphi_1 \sin \varphi_3 & \cos \varphi_1 \sin \varphi_2 \sin \varphi_3 + \sin \varphi_1 \cos \varphi_3 & \cos \varphi_1 \cos \varphi_2 \end{bmatrix} \]

\((c_2)\) it follows from rotation of \( \mathbf{e}_i \) around \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) that
\[ \mathbf{R} \rightarrow \mathbf{R} = R_3 R_2 R_1 \]
\[ = \underbrace{R_{3ij} R_{2jo} R_{1oj}}_{\mathbf{R}_{iq}} (\mathbf{e}_i \otimes \mathbf{e}_q) \]
Orthogonality of "CARDANO rotation tensors":

For all \( \mathbf{R} \in \{ \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}, \bar{\mathbf{R}} \} \), the following relations hold

\[
\mathbf{R}^{-1} = \mathbf{R}^T, \quad \text{i. e.} \quad \mathbf{R} \mathbf{R}^T = \mathbf{I} \quad \text{and} \quad (\det \mathbf{R})^2 = 1 \quad \rightarrow \quad \text{orthogonality}
\]

Furthermore, all rotation tensors hold the following relation

\[
\det \mathbf{R} = 1 : \quad \text{"proper" orthogonality}
\]

Rem.: A basis transformation with "non-proper" orthogonal transformations 
\((\det \mathbf{R} = -1)\) transforms a "right-handed" into a "left-handed" basis system.

Example:

here: Investigation of the orthogonality properties of 
\( \mathbf{R}_3 = \mathbf{R}_{3ij} (\mathbf{e}_i \otimes \mathbf{e}_j) \)

with 
\[
\mathbf{R}_{3ij} = \begin{bmatrix}
\cos \varphi_3 & -\sin \varphi_3 & 0 \\
\sin \varphi_3 & \cos \varphi_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

One looks at 
\[
\mathbf{R}_3 \mathbf{R}_3^T = \mathbf{R}_{3ij} (\mathbf{e}_i \otimes \mathbf{e}_j) \mathbf{R}_{3on} (\mathbf{e}_n \otimes \mathbf{e}_o) \\
= \mathbf{R}_{3ij} \mathbf{R}_{3on} \delta_{jn} (\mathbf{e}_i \otimes \mathbf{e}_o) = \mathbf{R}_{3in} \mathbf{R}_{3on} (\mathbf{e}_i \otimes \mathbf{e}_o)
\]

where 
\[
\mathbf{R}_{3in} \mathbf{R}_{3on} = \begin{bmatrix}
\sin^2 \varphi_3 + \cos^2 \varphi_3 & 0 & 0 \\
0 & \sin^2 \varphi_3 + \cos^2 \varphi_3 & 0 \\
0 & 0 & 1
\end{bmatrix} = \delta_{io}
\]

and one obtains 
\[
\mathbf{R}_3 \mathbf{R}_3^T = \delta_{io} (\mathbf{e}_i \otimes \mathbf{e}_o) = \mathbf{I} \quad \text{q. e. d.}
\]

Furthermore,

\[
\det \mathbf{R}_3 := \det (\mathbf{R}_{3ij}) = 1 \quad \rightarrow \quad \mathbf{R}_3 \text{ is proper orthogonal}
\]

Description of rotation tensors:

In general, the transformation between basis systems \( \mathbf{e}_i \) and basis systems \( \mathbf{e}^o_i \) satisfies the following relation:

\[
\mathbf{e}_i = \bar{\mathbf{R}} \mathbf{e}_i \quad \text{with} \quad \bar{\mathbf{R}} = \bar{\mathbf{R}}_{ik} \mathbf{e}_i \otimes \mathbf{e}_k
\]

\[
\rightarrow \quad \mathbf{e}_i = \bar{\mathbf{R}}^T \mathbf{e}^o_i \quad \text{with} \quad \bar{\mathbf{R}}^{-1} = \bar{\mathbf{R}}^T
\]
Otherwise,
\[ \tilde{e}_i = \tilde{R} \hat{e}_i \quad \text{with} \quad \tilde{R} = \tilde{R}_{ik} \hat{e}_i \otimes \hat{e}_k \]

**Consequence:** By comparing both relations, it follows that
\[ \tilde{R} = \tilde{R}^T, \quad \text{i.e.,} \quad \tilde{R}_{ik} \hat{e}_i \otimes \hat{e}_k = (\tilde{R}_{ik})^T \hat{e}_i \otimes \hat{e}_k \quad \rightarrow \quad \tilde{R}_{ik} = \tilde{R}_{ki} \]

In particular,
\[ \tilde{R} = \tilde{R}_{ik} (\hat{e}_i \otimes \hat{e}_k) = \tilde{R}_{ik} (\tilde{R} \hat{e}_i \otimes \tilde{R} \hat{e}_k) \]
\[ = \tilde{R}_{ik} \tilde{R}_{ni} \hat{e}_n \otimes \tilde{R}_{pk} \hat{e}_p = (\tilde{R}_{ni} \tilde{R}_{ik} \tilde{R}_{pk}) \hat{e}_n \otimes \hat{e}_p = \tilde{R}_{pn} \hat{e}_n \otimes \hat{e}_p = \tilde{R}^T \]
\[ \quad \rightarrow \quad \tilde{R}_{mi} \tilde{R}_{ik} \tilde{R}_{pk} = \tilde{R}_{pn} \iff \tilde{R}_{mi} \tilde{R}_{ik} = \delta_{nk} \]

**Rem.:** The coefficient matrices \( \tilde{R}_{ni} \) and \( \tilde{R}_{ik} \) are inverse to each other, i.e., in general, \( \tilde{R}_{ni} \tilde{R}_{ik} = \delta_{nk} \) implies 6 equations for the 9 unknown coefficients \( \tilde{R}_{ik} \). Due to \( \tilde{R}^{-1} = \tilde{R}^T \), one has \( \tilde{R}_{ni}^{-1} = (\tilde{R}_{ni})^T = \tilde{R}_{ni} \), i.e.
\[ \tilde{R}_{ik} = (\tilde{R}_{ik})^T = \tilde{R}_{ki} \]

(C) **Introduction of Euler Angles**

**Rem.:** Rotation of a basis system \( e_i \) around three specific axes.

Introduction of 3 specific angles around \( e_3, \tilde{e}_1, \tilde{e}_3 = \tilde{e}_3 \)

**Illustration:**

**Idea:** Given are 2 planes \( F \) and \( \hat{F} \) with in-plane vectors \( e_1, e_2 \) and \( \tilde{e}_1, \tilde{e}_2 \) and surface normals \( e_3 \) and \( \tilde{e}_3 \). The basis systems \( e_i \) and \( \tilde{e}_i \) are related to each other by the Eulerian rotation tensor \( R \):
\[ \tilde{e}_i := R e_i \]
1st step:

\[ \bar{\mathbf{e}}_3 = \mathbf{e}_3 \]

Rotation of \( \mathbf{e}_i \) in plane \( \mathcal{F} \) around \( \mathbf{e}_3 \) with the angle \( \varphi \), such that \( \bar{\mathbf{e}}_i \) is directed towards \( c{-}c \). This yields the rotation tensor

\[
\mathbf{R}_3 = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Then, the new system \( \bar{\mathbf{e}}_i \) is computed as follows

\[
\bar{\mathbf{e}}_i = \mathbf{R}_3 \mathbf{e}_i = R_{3jk} (\mathbf{e}_j \otimes \mathbf{e}_k) \mathbf{e}_i = R_{3ji} \mathbf{e}_j .
\]

Thus,

\[
\bar{\mathbf{e}}_1 = R_{3j1} \mathbf{e}_j = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2 \\
\bar{\mathbf{e}}_2 = R_{3j2} \mathbf{e}_j = -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2 \\
\bar{\mathbf{e}}_3 = R_{3j3} \mathbf{e}_j = \mathbf{e}_3 .
\]

2nd step:

Rotation of \( \bar{\mathbf{e}}_i \) around \( \bar{\mathbf{e}}_1 \) with the angle \( \delta \), such that \( \bar{\mathbf{e}}_2 \) lies in the plane \( \bar{\mathcal{F}} \), and \( \bar{\mathbf{e}}_3 \) is directed normal to the plane \( \bar{\mathcal{F}} \). This yields the rotation tensor

\[
\bar{\mathbf{R}}_1 = \begin{bmatrix} 
1 & 0 & 0 \\
0 & \cos \delta & -\sin \delta \\
0 & \sin \delta & \cos \delta
\end{bmatrix}
\]

Then, the new system \( \bar{\mathbf{e}}_i \) is computed as follows

\[
\bar{\mathbf{e}}_i = \bar{\mathbf{R}}_1 \bar{\mathbf{e}}_i = \bar{R}_{1jk} (\bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k) \bar{\mathbf{e}}_i = \bar{R}_{1ji} \bar{\mathbf{e}}_j .
\]

Thus,

\[
\bar{\mathbf{e}}_1 = \bar{R}_{1j1} \bar{\mathbf{e}}_j = \bar{\mathbf{e}}_1 \\
\bar{\mathbf{e}}_2 = \bar{R}_{1j2} \bar{\mathbf{e}}_j = \cos \delta \bar{\mathbf{e}}_2 + \sin \delta \bar{\mathbf{e}}_3 \\
\bar{\mathbf{e}}_3 = \bar{R}_{1j3} \bar{\mathbf{e}}_j = -\sin \delta \bar{\mathbf{e}}_2 + \cos \delta \bar{\mathbf{e}}_3 .
\]

3rd step:

Rotation of \( \bar{\mathbf{e}}_i \) in plane \( \bar{\mathcal{F}} \) around \( \bar{\mathbf{e}}_3 \) with the angle \( \psi \). This yields the rotation tensor

\[
\bar{\mathbf{R}}_3 = \begin{bmatrix} 
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Then, the new system \( \bar{\mathbf{e}}_i \) is computed as follows

\[
\bar{\mathbf{e}}_i = \bar{\mathbf{R}}_3 \bar{\mathbf{e}}_i = \bar{R}_{3jk} (\bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k) \bar{\mathbf{e}}_i = \bar{R}_{3ji} \bar{\mathbf{e}}_j .
\]
Thus,
\[\begin{align*}
\dot{\mathbf{e}}_1 &= \mathbf{R}_{3j1} \mathbf{e}_j = \cos \psi \mathbf{e}_1 + \sin \psi \mathbf{e}_2 \\
\dot{\mathbf{e}}_2 &= \mathbf{R}_{3j2} \mathbf{e}_j = -\sin \psi \mathbf{e}_1 + \cos \psi \mathbf{e}_2 \\
\dot{\mathbf{e}}_3 &= \mathbf{R}_{3j3} \mathbf{e}_j = \mathbf{e}_3.
\end{align*}\]

**Summary:**

(a) Inserting \(\dot{\mathbf{e}}_i = \mathbf{R}_1 \mathbf{e}_i\)
\[\begin{align*}
\dot{\mathbf{e}}_1 &= \cos \psi \mathbf{e}_1 + \sin \psi (\cos \delta \mathbf{e}_2 + \sin \delta \mathbf{e}_3) \\
\dot{\mathbf{e}}_2 &= -\sin \psi \mathbf{e}_1 + \cos \psi (\cos \delta \mathbf{e}_2 + \sin \delta \mathbf{e}_3) \\
\dot{\mathbf{e}}_3 &= -\sin \delta (\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) + \cos \delta \mathbf{e}_3
\end{align*}\]

\[\rightarrow \quad \dot{\mathbf{e}}_i = \mathbf{R}_3 (\mathbf{R}_1 \mathbf{e}_i) =: \mathbf{R} \mathbf{e}_i \quad \text{with} \quad \mathbf{R} = \mathbf{R}_3 \mathbf{R}_1 \]

(b) Inserting \(\dot{\mathbf{e}}_i = \mathbf{R}_3 \mathbf{e}_i\)
\[\begin{align*}
\dot{\mathbf{e}}_1 &= \cos \psi (\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2) + \sin \psi \cos \delta (-\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) + \sin \psi \sin \delta \mathbf{e}_3 \\
\dot{\mathbf{e}}_2 &= -\sin \psi (\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2) + \cos \psi \cos \delta (-\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) + \cos \psi \sin \delta \mathbf{e}_3 \\
\dot{\mathbf{e}}_3 &= -\sin \delta (-\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) + \cos \delta \mathbf{e}_3
\end{align*}\]

\[\rightarrow \quad \dot{\mathbf{e}}_i = \mathbf{R} (\mathbf{R}_3 \mathbf{e}_i) =: \mathbf{R} \mathbf{e}_i \quad \text{with} \quad \mathbf{R} = \mathbf{R}_3 \mathbf{R}_1 \mathbf{R}_3 \]

Rotation tensors \(\mathbf{R}\) and \(\dot{\mathbf{R}}\):

For the total rotation the following relation holds:
\[\begin{align*}
\dot{\mathbf{e}}_i &= (\mathbf{R}_3 \mathbf{R}_1 \mathbf{R}_3) \mathbf{e}_i =: \mathbf{R} \mathbf{e}_i \\
&= (\mathbf{R}_3 \mathbf{R}_1) (\mathbf{R}_3 \mathbf{e}_i) = \mathbf{R}_3 (\mathbf{R}_1 \mathbf{e}_i) = \mathbf{R}_3 \mathbf{e}_i = \mathbf{R} \mathbf{e}_i
\end{align*}\]
Furthermore,\[ \mathbf{e}_i = \mathbf{R} \mathbf{e}_i \quad \rightarrow \quad \mathbf{e}_i = \mathbf{R}^T \mathbf{e}_i = : \mathbf{R}^* \mathbf{e}_i \quad \rightarrow \quad \mathbf{R} = \mathbf{R}^T \]

Analogously to the previous considerations \[ R_{ik} = (R_{ik})^T = R_{ki} \]

Description:

\[
\mathbf{R} = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
\sin \psi \sin \delta & \cos \psi \cos \delta & -\sin \delta
\end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_k
\]

Combining rotation tensors with different basis systems:

Example: \( \mathbf{R} := \hat{\mathbf{R}}_3 \hat{\mathbf{R}}_1 \)

\[
\mathbf{e}_i' = \hat{\mathbf{R}}_3 \hat{\mathbf{R}}_1 \mathbf{e}_i = (\hat{\mathbf{R}}_3 \hat{\mathbf{R}}_1) \mathbf{e}_i
\]

\[
\rightarrow \quad \mathbf{R} = \hat{R}_{3ik} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_k) \hat{R}_{1no} (\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_o)
\]

\[
= \hat{R}_{3ik} (\hat{\mathbf{R}}_1 \hat{\mathbf{e}}_i \otimes \hat{\mathbf{R}}_1 \hat{\mathbf{e}}_k) \hat{R}_{1no} (\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_o)
\]

\[
= \hat{R}_{1si} \hat{R}_{3ik} \hat{R}_{1tk} \hat{R}_{1no} (\hat{\mathbf{e}}_s \otimes \hat{\mathbf{e}}_o)
\]

Thus, the rotation tensor \( \mathbf{R} \) is given by

\[
\mathbf{R} = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
\sin \psi \sin \delta & \cos \psi \cos \delta & -\sin \delta
\end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_k
\]

Rem.: Concerning CARDANO angles, all partial rotations (e. g. \( \mathbf{R} = \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1 \) with \( \mathbf{e}_i' = \mathbf{R} \mathbf{e}_i \)) are carried out with respect to the same basis \( \mathbf{e}_i \), i. e. the combination of the partial rotations is much easier.

Rotation around a fixed axis:

Rem.: A rotation around 3 independent axes can also be described by a rotation around the resulting axis of rotation:

\[ \rightarrow \quad \text{EULER-RODRIGUES representation of the spatial rotation} \]

The EULER-RODRIGUES representation of the rotation is discussed later (see section 2.7).
2.5 Higher order tensors

**Definition:** An arbitrary $n$-th order tensor is given by

$$\mathbf{A} \in \mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 \quad (n \text{ times})$$

with $\mathcal{V}^3 \otimes \mathcal{V}^3 \otimes \cdots \otimes \mathcal{V}^3 : n$-th order dyadic product space

**Rem.:** Usually, $n \geq 2$. However, there exist special cases for $n = 1$ (vector) and $n = 0$ (scalar).

**General description of the linear mapping**

**Definition:** A linear mapping is a “contracting product” (contraction) given by

$$\mathbf{A} \mathbf{B} = \mathbf{C} \quad \text{with} \quad n \geq s$$

Descriptive example on simple tensors:

$$\begin{align*}
\left(\text{a} \otimes \text{b} \otimes \text{c} \otimes \text{d}\right) \left(\text{e} \otimes \text{f}\right) &= \left(\text{c} \cdot \text{e}\right) \left(\text{d} \cdot \text{f}\right) \text{a} \otimes \text{b} \\
\begin{array}{c}
\text{A} \\
\text{B} \\
\text{C}
\end{array}
\end{align*}$$

**Fundamental 4-th order tensors**

**Rem.:** 4-th order fundamental tensors are built by a dyadic product of 2nd order identity tensors and the corresponding independent transpositions.

One introduces:

$$\begin{align*}
\mathbf{I} \otimes \mathbf{I} &= (\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_j \otimes \mathbf{e}_j) \\
(\mathbf{I} \otimes \mathbf{I})^{23} &= \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j \\
(\mathbf{I} \otimes \mathbf{I})^{24} &= \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i
\end{align*}$$

with $(\cdot)^T$: transposition, defined by the exchange of the $i$-th and the $k$-th basis system

**Rem.:** Further transpositions of $\mathbf{I} \otimes \mathbf{I}$ do not lead to further independent tensors. The fundamental tensors from above exhibit the property

$$\begin{align*}
\mathbf{A}^4 &= \mathbf{A}^{4T} \quad \text{with} \quad \mathbf{A}^{4} = \left(\mathbf{A}^{T}\right)^{24}
\end{align*}$$

**Consequence:** The 4-th order fundamental tensors are symmetric (concerning an exchange of the first two and the second two basis systems).
Properties of 4-th order fundamental tensors

(a) identical map
\[
(I \otimes I)^{23} A = (e_i \otimes e_j \otimes e_i \otimes e_j) a_{st} (e_s \otimes e_t) \\
= a_{st} \delta_{is} \delta_{jt} (e_i \otimes e_j) = a_{ij} (e_i \otimes e_j) = A
\]
\[\rightarrow 4 \mathbf{I} := (I \otimes I)^{23} \text{ is 4-th order identity tensor}\]

(b) “transposing” map
\[
(I \otimes I)^{24} A = (e_i \otimes e_j \otimes e_j \otimes e_i) a_{st} (e_s \otimes e_t) \\
= a_{st} \delta_{is} \delta_{jt} (e_i \otimes e_j) = a_{ji} (e_i \otimes e_j) = A^T
\]

(c) “tracing” map
\[
(I \otimes I) A = (e_i \otimes e_i \otimes e_j \otimes e_j) a_{st} (e_s \otimes e_t) \\
= a_{st} \delta_{is} \delta_{jt} (e_i \otimes e_i) = a_{jj} (e_i \otimes e_i)
\]
\[= (A \cdot I) I = (\text{tr} A) I\]
\[
\text{with } A \cdot I = a_{st} (e_s \otimes e_t) \cdot (e_j \otimes e_j) = a_{st} \delta_{sj} \delta_{tj} = a_{jj}
\]

Specific 4-th order tensors

Let \(A, B, C, D\) be arbitrary 2nd order tensors. Then, a 4-th order tensor \(\mathbf{A}\) can be defined exhibiting the following properties:
\[
\mathbf{A} = (A \otimes B)^{23} = (B^T \otimes A^T)^{24} \quad (\ast)
\]
\[
\mathbf{A}^T = [(A \otimes B)^{23}]^T = (A^T \otimes B^T)^{23}
\]
\[
\mathbf{A}^{-1} = [(A \otimes B)^{23}]^{-1} = (A^{-1} \otimes B^{-1})^{23}
\]

Furthermore, following relation holds:
\[
(\cdot)^T = [(\cdot)^{13}]^{24}
\]

From (\ast), the following relations can be derived:
\[
(A \otimes B)^{23} (C \otimes D)^{23} = (AC \otimes BD)^{23}
\]
\[
(A \otimes B)^{23} (C \otimes D) = (ACB^T \otimes D)
\]
\[
(A \otimes B)(C \otimes D)^{23} = (A \otimes C^T BD)
\]

and
\[
(A \otimes B)^{23} C = ACB^T
\]
\[
(A \otimes B)^{23} \mathbf{v} = [A \otimes (B \mathbf{v})]^{23}
\]
Defining a 4-th order tensor $\mathbf{B}$ with the properties

$$
\mathbf{B}^4 = (A \otimes B)^{24} = [(A \otimes B)^{13}]^T
$$

$$
\mathbf{B}^T = [(A \otimes B)^{23}]^T = (B \otimes A)^{34}
$$

$$
\mathbf{B}^{-1} = [(A \otimes B)^{23}]^{-1} = (B^{T-1} \otimes A^{T-1})^{23}
$$

it can be shown that

$$
(A \otimes B)^{24} (C \otimes D)^{24} = (AD^T \otimes B^T C)^{23}
$$

$$
(A \otimes B)^{23} (C \otimes D)^{24} = (AC \otimes DB^T)^{24}
$$

$$
(A \otimes B)^{24} (C \otimes D)^{23} = (AD \otimes C^T B)^{24}
$$

$$
(A \otimes B)(C \otimes D)^{24} = (A \otimes DB^T C)
$$

and

$$
(A \otimes B)^{24} C = AC^T B
$$

Furthermore, the following relation holds:

$$
(\mathbf{C} \mathbf{D})^T = \mathbf{D}^T \mathbf{C}^T
$$

where $\mathbf{C}$ and $\mathbf{D}$ are arbitrary 4-th order tensors.

**High order tensors and incomplete mappings**

If higher order tensors are applied to other tensors in the sense of incomplete mappings, one has to know how many of the basis vectors have to be linked by scalar products. Therefore, a underlined superscript $(\cdot)^\perp$ indicates the order of the desired result after the tensor operation has been carried out.

**Examples in basis notation:**

$$
(\mathbf{A} \mathbf{B})^\perp = [a_{ijkl} (e_i \otimes e_j \otimes e_k \otimes e_l) b_{mno} (e_m \otimes e_n \otimes e_o)]^\perp
$$

$$
= a_{ijkl} b_{mno} \delta_{km} \delta_{ln} (e_i \otimes e_j \otimes e_o)
$$

$$
(\mathbf{A} \mathbf{B})^\perp = [a_{ij} (e_i \otimes e_j) b_{mno} (e_m \otimes e_n \otimes e_o)]^\perp
$$

$$
= a_{ij} b_{mno} \delta_{im} \delta_{jn} e_o
$$

**Note:** Note in passing that the incomplete mapping is governed by scalar products of a sufficient number of inner basis systems.
2.6 Fundamental tensor of 3rd order (Ricci permutation tensor)

**Rem.:** The fundamental tensor of 3rd order is introduced in the context of the “outer product” (e. g. vector product between vectors).

**Definition:** The fundamental tensor $^{3}_{E}$ satisfies the rule

$$u \times v = ^{3}_{E} (u \otimes v)$$

**Introduction of $^{3}_{E}$ in basis notation:**

There is

$$^{3}_{E} = e_{ijk} (e_{i} \otimes e_{j} \otimes e_{k})$$

with the “permutation symbol” $e_{ijk}$

$$e_{ijk} = \begin{cases} 
1 & : \text{even permutation} \\
-1 & : \text{odd permutation} \\
0 & : \text{double indexing} 
\end{cases} \rightarrow \begin{cases} 
\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\
\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\
\text{all remaining } e_{ijk} \text{ vanish} 
\end{cases}$$

**Application of $^{3}_{E}$ to the vector product of vectors:**

From the above definition,

$$u \times v = ^{3}_{E} (u \otimes v)$$

$$= e_{ijk} (u_{i} \otimes v_{j} \otimes e_{k}) (u_{s} e_{s} \otimes v_{t} e_{t})$$

$$= e_{ijk} u_{s} v_{t} \delta_{js} \delta_{kt} e_{i} = e_{ijk} u_{j} v_{k} e_{i}$$

$$= (u_{2} v_{3} - u_{3} v_{2}) e_{1} + (u_{3} v_{1} - u_{1} v_{3}) e_{2} + (u_{1} v_{2} - u_{2} v_{1}) e_{3}$$

Comparison with the computation by use of the matrix notation, cf. page 5

$$u \times v = \begin{vmatrix} 
e_{1} & e_{2} & e_{3} \\
u_{1} & u_{2} & u_{3} \\
v_{1} & v_{2} & v_{3} 
\end{vmatrix} = \cdots \text{ q. e. d.}$$

**An identity for $^{3}_{E}$:**

Incomplete mapping of two Ricci-tensors yielding a 2nd or 4th order object

$$^{3}_{E}^{3}_{E} = 2I, \quad ^{3}_{E}^{3}_{E} = (I \otimes I)^{T} - (I \otimes I)^{24}$$

2.7 The axial vector

**Rem.:** The axial vector (pseudo vector) can be used for the description of rotations (rotation vector).

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Definition: The axial vector $\mathbf{t}$ is associated with the skew-symmetric part $\text{skw} \mathbf{T}$ of an arbitrary tensor $\mathbf{T} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ via

$$\mathbf{t} := \frac{1}{2} \mathbf{E} \mathbf{T}^T$$

One calculates,

$$\mathbf{t} = \frac{1}{2} \epsilon_{ijk} \left( e_i \otimes e_j \otimes e_k \right) t_{st} \left( e_t \otimes e_s \right)$$

$$= \frac{1}{2} \epsilon_{ijk} t_{st} \delta_{jt} \delta_{ks} e_i = \frac{1}{2} \epsilon_{ijk} t_{kj} e_i$$

$$= \frac{1}{2} \left[ (t_{32} - t_{23}) e_1 + (t_{13} - t_{31}) e_2 + (t_{21} - t_{12}) e_3 \right]$$

It follows from 2.3 (b)

$$\mathbf{T} = \text{sym} \mathbf{T} + \text{skw} \mathbf{T}$$

Thus, the axial vector of $\mathbf{T}$ is given by

$$\mathbf{t} = \frac{1}{2} \mathbf{E} \left( \text{sym} \mathbf{T} + \text{skw} \mathbf{T} \right)^T$$

$$= \frac{1}{2} \mathbf{E} \left( \text{skw} \mathbf{T}^T \right) = -\frac{1}{2} \mathbf{E} \left( \text{skw} \mathbf{T} \right)$$

Rem.: A symmetric tensor has no axial vector.

Axial vector and linear mapping:

The following relation holds:

$$(\text{skw} \mathbf{T}) \mathbf{v} = \mathbf{t} \times \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{V}^3$$

Axial vector and the vector product of tensors:

Definition: The vector product of 2 tensors $\{\mathbf{T}, \mathbf{S}\} \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$\mathbf{S} \times \mathbf{T} = \mathbf{E} \left( \mathbf{S} \mathbf{T}^T \right)$$

Rem.: The vector product (cross product) of 2 tensors yields a vector.

In comparison with the definition of the axial vector follows

$$\mathbf{I} \times \mathbf{T} = \frac{1}{2} \mathbf{E} \mathbf{T}^T = 2 \mathbf{t}$$

Furthermore, the vector product of 2 tensors yields

$$\mathbf{S} \times \mathbf{T} = -\mathbf{T} \times \mathbf{S}$$
Axial vector and outer tensor product of vector and tensor:

**Definition:** The outer tensor product of a vector $u \in \mathcal{V}^3$ and a tensor $T \in \mathcal{V}^3 \otimes \mathcal{V}^3$ satisfies

$$(u \times T)v = u \times (Tv) ; \quad v \in \mathcal{V}^3$$

**Rem.:** The outer tensor product of vector and tensor yields a tensor.

The following relations hold:

$$u \times T = -(u \times T)^T = -T \times u$$

$\rightarrow$ i. e. $u \times T$ is skew-symmetric

$$u \times T = [\frac{3}{2} \mathbf{E} (u \otimes T)^2]$$

with $(\cdot)^2$ : “incomplete” linear mapping (association) resulting in a 2nd order tensor.

Evaluation in basis notation leads to

$$u \times T = [(e_{ijk} e_i \otimes e_j \otimes e_k) (u_r e_r \otimes t_{st} e_s \otimes e_t)]^2$$

$$= e_{ijk} u_r t_{st} \delta_{jr} \delta_{ks} (e_i \otimes e_t)$$

$$= e_{ijk} u_j t_{kt} (e_i \otimes e_t)$$

In particular, if $T \equiv I$, the following relation holds:

$$u \times I = [\frac{3}{2} \mathbf{E} (u \otimes I)^2] = e_{ijk} u_j \delta_{kt} (e_i \otimes e_t) = e_{ijt} u_j (e_i \otimes e_t)$$

Furthermore, for the special tensor $u \times I$ follows

$$\frac{3}{2} \mathbf{E} (u \times I) = -2u$$

$\rightarrow$ $u = -\frac{1}{2} \frac{3}{2} \mathbf{E} (u \times I) = \frac{1}{2} \frac{3}{2} \mathbf{E} (u \times I)^T$

**Consequence:** In the tensor $u \times I$, the vector $u$ is already the corresponding axial vector.

Finally, the following relation holds:

$$u \times I = - \frac{3}{2} \mathbf{E} u$$

$$\rightarrow \frac{3}{2} \mathbf{E} (u \times I) = - \frac{3}{2} \mathbf{E} \frac{3}{2} \mathbf{E} u = -(\mathbf{E} \mathbf{E})^2 u \overset{!}{=} -2u$$

i. e. $(\mathbf{E} \mathbf{E})^2 = 2I$
Some additional rules:

\[(a \times b) \otimes c = a \times (b \otimes c)\]

\[(I \times T) \cdot w = T \cdot \Omega \quad \text{with} \quad \Omega = w \times I\]

**APPLICATION TO THE TENSOR PRODUCT OF VECTOR AND TENSOR**

**Rotation around a fixed spatial axis**

\[
\begin{align*}
\text{Rotation of } x \text{ around axis } e
\end{align*}
\]

\[
\begin{align*}
x &= a + \hat{u} = a + C_1 u + b \\
\text{with } &\begin{cases} 
  a = (x \cdot e) e \\
  u = x - a \\
  b = C_2 (e \times x)
\end{cases} \\
\text{and } &\varphi = \varphi e; \quad |e| = 1
\end{align*}
\]

**Determination of the constants** \(C_1\) and \(C_2\):

(a) For the angle between \(u\) and \(\hat{u}\), the following relation holds

\[
\cos \varphi = \frac{u \cdot \hat{u}}{|u||\hat{u}|} \quad \text{with} \quad |u| = |\hat{u}|
\]

Furthermore, the following relation holds

\[
u \cdot \hat{u} = u \cdot (C_1 u + b) = C_1 u \cdot u + \underbrace{u \cdot b}_{= 0, \text{ as } u \perp b} = C_1 |u|^2
\]

Thus,

\[
\cos \varphi = \frac{C_1 |u|^2}{|u|^2} = C_1 \quad \rightarrow \quad C_1 = \cos \varphi
\]

(b) For the angle between \(b\) and \(\hat{u}\), the following relation holds

\[
\cos(90^\circ - \varphi) = \sin \varphi = \frac{b \cdot \hat{u}}{|b||\hat{u}|}
\]

Furthermore, the following relation holds

\[
b \cdot \hat{u} = b \cdot (C_1 u + b) = C_1 b \cdot u + b \cdot b = |b|^2
\]

Thus,

\[
= 0, \text{ as } u \perp b
\]

and

\[
|b| = C_2 |e \times x| = C_2 |e| \frac{|x| \sin \varphi(e; x)}{|u|} = C_2 |u|
\]
Thus, leading to
\[
\sin \varphi = \frac{|b|^2}{|b||u|} = \frac{|b|}{|u|} = C_2 \frac{|u|}{|u|} = C_2 \quad \rightarrow \quad C_2 = \sin \varphi
\]

Thus, \( \mathbf{x} \) is given by
\[
\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}) \mathbf{e} + \cos \varphi [\mathbf{x} - (\mathbf{x} \cdot \mathbf{e}) \mathbf{e}] + \sin \varphi (\mathbf{e} \times \mathbf{x})
\]

**Determination of the rotation tensor \( \mathbf{R} \):**

For the tensor product of vector and tensor, the following relation holds:
\[
(\mathbf{e} \times \mathbf{I}) \mathbf{x} = \mathbf{e} \times (\mathbf{I} \mathbf{x}) = \mathbf{e} \times \mathbf{x}
\]

Thus,
\[
\mathbf{x} = (\mathbf{x} \otimes \mathbf{e}) \mathbf{x} + \cos \varphi (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \mathbf{x} + \sin \varphi (\mathbf{e} \times \mathbf{I}) \mathbf{x} = \mathbf{R} \mathbf{x}
\]

\[
\mathbf{R} = \mathbf{e} \otimes \mathbf{e} + \cos \varphi (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) + \sin \varphi (\mathbf{e} \times \mathbf{I})
\]

\( (*) \)

**Rem.:** \( (*) \) is the **Euler-Rodrigues** form of the spatial rotation.

**Example:** Rotation with \( \varphi_3 \) around the \( \mathbf{e}_3 \) axis
\[
\mathbf{R} = \mathbf{R}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3 + \cos \varphi_3 (\mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3) + \sin \varphi_3 (\mathbf{e}_3 \times \mathbf{I})
\]

The following relation holds:
\[
\mathbf{e}_3 \times \mathbf{I} = \left[ \mathbf{E} (\mathbf{e}_3 \otimes \mathbf{I}) \right]^2
\]
\[
= \left[ e_{ijk} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) (\mathbf{e}_3 \otimes \mathbf{e}_l \otimes \mathbf{e}_m) \right]^2
\]
\[
= e_{ijk} \delta_{j3} \delta_{kl} (\mathbf{e}_i \otimes \mathbf{e}_l) = e_{ilj} (\mathbf{e}_i \otimes \mathbf{e}_l)
\]
\[
= \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2
\]

Thus, leading to
\[
\mathbf{R}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3 + \cos \varphi_3 (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin \varphi_3 (\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2)
\]
\[
= \mathbf{R}_{3ij} (\mathbf{e}_i \otimes \mathbf{e}_j)
\]

with
\[
\mathbf{R}_{3ij} = \begin{bmatrix}
\cos \varphi_3 & - \sin \varphi_3 & 0 \\
\sin \varphi_3 & \cos \varphi_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

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2.8 The outer tensor product of tensors

Definition: The outer tensor product of tensors (double cross product) is defined via

\[(A \times B)(u_1 \times u_2) := Au_1 \times Bu_2 - Au_2 \times Bu_1\]

As a direct consequence, one finds

\[A \times B = B \times A\]

Furthermore, the following relations hold:

\[(A \times B)^T = A^T \times B^T\]
\[(A \times B)(C \times D) = (A \times C \times B \times D) + (A \times D \times B \times C)\]
\[(I \times I) = 2I\]
\[(a \otimes b) \times (c \otimes d) = (a \times c) \otimes (b \times d)\]
\[(A \times B) \cdot C = (B \times C) \cdot A = (C \times A) \cdot B\]

From the above definition, it is easily proved that

\[[(A \times B) \cdot C][u_1 \times u_2] \cdot u_3 = e_{ijk} (Au_i \times Bu_j) \cdot Cu_k\]

The outer tensor product in basis notation

\[A \times B = a_{ik} (e_i \otimes e_k) \otimes b_{no} (e_n \otimes e_o)\]
\[= a_{ik} b_{no} (e_i \times e_n) \otimes (e_k \times e_o)\]

with \[e_i \times e_n = \frac{3}{E} (e_i \otimes e_n) = e_{ijn} e_j\]
\[e_k \times e_o = \frac{3}{E} (e_k \otimes e_o) = e_{kop} e_p\]

\[\rightarrow A \times B = a_{ik} b_{no} e_{ijn} e_{kop} (e_j \otimes e_p)\]

Furthermore, it follows that

\[A \times I = (A \cdot I)I - A^T\]
\[A \times B = (A \cdot I)(B \cdot I)I - (A^T \cdot B)I - (A \cdot I)B^T - (B \cdot I)A^T + A^T B^T + B^T A^T\]
\[(A \times B) \cdot C = (A \cdot I)(B \cdot I)(C \cdot I) - (A \cdot I)(B^T \cdot C) - (B \cdot I)(A^T \cdot C) - (C \cdot I)(A^T \cdot B) + (A^T B^T) \cdot C + (B^T A^T) \cdot C\]

The cofactor, the adjoint tensor and the determinant:
The following relations hold:

\[
\text{cof} \mathbf{A} = \frac{1}{2} \mathbf{A} \times \mathbf{A} =: \mathbf{A}^+, \quad \text{adj} \mathbf{A} = (\text{cof} \mathbf{A})^T
\]

\[
\det \mathbf{A} = \frac{1}{6} (\mathbf{A} \times \mathbf{A}) \cdot \mathbf{A} = \det |a_{ik}| = \frac{(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3}{(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3}
\]

In basis notation the following relation holds:

\[
\mathbf{A}^+ = \frac{1}{2} (a_{ik} a_{no} e_{inj} e_{kop}) (e_j \otimes e_p) = a_{jp} (e_j \otimes e_p)
\]

**Rem.:** The coefficient matrix \( a_{jp} \) of the cofactor \( \text{cof} \mathbf{A} \) contains at each position \( (\cdot)_{jp} \) the corresponding subdeterminant of \( \mathbf{A} \)

\[
a_{11} = a_{22} a_{33} - a_{23} a_{32} \quad \text{etc.}
\]

**The inverse tensor:**

The following relation holds:

\[
\mathbf{A}^{-1} = (\det \mathbf{A})^{-1} \text{adj} \mathbf{A} ; \quad \mathbf{A}^{-1} \text{ exists if } \det \mathbf{A} \neq 0
\]

**Rules for the cofactor, the determinant and the inverse tensor:**

\[
\begin{align*}
\det (\mathbf{A} \mathbf{B}) &= \det \mathbf{A} \det \mathbf{B} \\
\det (\alpha \mathbf{A}) &= \alpha^3 \det \mathbf{A} \\
\det \mathbf{I} &= 1 \\
\det \mathbf{A}^T &= \det \mathbf{A} \\
\det \mathbf{A}^+ &= (\det \mathbf{A})^2 \\
\det \mathbf{A}^{-1} &= (\det \mathbf{A})^{-1} \\
\det (\mathbf{A} + \mathbf{B}) &= \det \mathbf{A} + \mathbf{A}^+ \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{B}^+ + \det \mathbf{B} \\
(\mathbf{A} \mathbf{B})^+ &= \mathbf{A}^+ \mathbf{B}^+ \\
(\mathbf{A}^+)^T &= (\mathbf{A}^T)^+
\end{align*}
\]

**2.9 The eigenvalue problem and the invariants of tensors**

**Definition:** The eigenvalue problem of an arbitrary 2nd order tensor \( \mathbf{A} \) is given by

\[
(\mathbf{A} - \gamma_{\mathbf{A}} \mathbf{I}) \mathbf{a} = 0 , \quad \text{where} \begin{cases} 
\gamma_{\mathbf{A}} : \text{eigenvalue} \\
\mathbf{a} : \text{eigenvector}
\end{cases}
\]
Formal solution for $a$ yields

$$a = (A - \gamma_{A} I)^{-1} 0 = \text{adj} (A - \gamma_{A} I) \frac{0}{\det (A - \gamma_{A} I)}$$

**Consequence:** Non-trivial solution for $a$ only if the characteristic equation is fulfilled, i.e.

$$\det (A - \gamma_{A} I) = 0$$

With the determinant rule

$$\det (A + B) = \frac{1}{6} [(A + B) \times (A + B)] \cdot (A + B)$$

$$= \frac{1}{6} (A \times A) \cdot A + \frac{1}{6} (A \times A) \cdot B + \frac{1}{3} (A \times B) \cdot A +$$

$$+ \frac{1}{3} (A \times B) \cdot B + \frac{1}{6} (B \times B) \cdot A + \frac{1}{6} (B \times B) \cdot B$$

follows

$$\det (A - \gamma_{A} I) = \det A + \hat{A} \cdot B + A \cdot \hat{B} + \det B$$

the characteristic equation can be simplified to

$$\det (A - \gamma_{A} I) = III_{A} - \gamma_{A} II_{A} + \gamma_{A}^{2} I_{A} - \gamma_{A}^{3} = 0$$

**Rem.:** The abbreviations $I_{A}$, $II_{A}$ and $III_{A}$ are the three scalar principal invariants of a tensor $A$ which play an important role in the field of continuum mechanics.

**Alternative representations of the principal invariants**

Scalar product representation:

$$I_{A} = A \cdot I = \text{tr} A$$

$$II_{A} = \frac{1}{2} (I_{A}^{2} - A A \cdot I) = \frac{1}{2} [(\text{tr} A)^{2} - \text{tr} (A A)]$$

$$III_{A} = \frac{1}{6} I_{A}^{3} - \frac{1}{2} I_{A}^{2} (A A \cdot I) + \frac{1}{2} A^{T} A^{T} \cdot A =$$

$$= \frac{1}{6} [(\text{tr} A)^{3} - 3 \text{tr} A \text{tr} (A A) + 2 \text{tr} (A A A)] = \det A$$

Eigenvalue representation:

$$I_{A} = \gamma_{A(1)} + \gamma_{A(2)} + \gamma_{A(3)}$$

$$II_{A} = \gamma_{A(1)} \gamma_{A(2)} + \gamma_{A(2)} \gamma_{A(3)} + \gamma_{A(3)} \gamma_{A(1)}$$

$$III_{A} = \gamma_{A(1)} \gamma_{A(2)} \gamma_{A(3)}$$

**CALEY-HAMILTON-Theorem:**

$$A A A - I_{A} A A + II_{A} A - III_{A} = 0$$
3 Fundamentals of vector and tensor analysis

3.1 Introduction of functions

Notation:

\[
\begin{align*}
\phi(\cdot) & : \text{scalar-valued function} \\
\text{exists} & \left\{ v(\cdot) : \text{vector-valued function} \right\} \\
T(\cdot) & : \text{tensor-valued function}
\end{align*}
\]

of \((\cdot)\)

Notation:

- **Domain** of a function: set of all possible values of the independent variable quantities (variables); usually contiguous
- **Range** of a function: set of all possible values of the dependent variable quantities: \(\phi(\cdot); \ v(\cdot); \ T(\cdot)\)

Example: \(\phi(A)\) : scalar-valued tensor function

Notions:

- **Domain** of a function: set of all possible values of the independent variable quantities (variables); usually contiguous
- **Range** of a function: set of all possible values of the dependent variable quantities: \(\phi(\cdot); \ v(\cdot); \ T(\cdot)\)

3.2 Functions of scalar variables

Here: Vector- and tensor-valued functions of real scalar variables

(a) **Vector-valued functions of a single variable**

It exists:

\[
u = u(\alpha) \quad \text{with} \quad \left\{ \begin{array}{l}
u : \text{unique vector-valued function,} \\
\text{range in the open domain } V^3 \\
\alpha : \text{real scalar variable}
\end{array} \right. \]

Derivative of \(u(\alpha)\) with the differential quotient:

\[
\mathbf{w}(\alpha) := u'(\alpha) := \frac{d u(\alpha)}{d \alpha}
\]

Differential of \(u(\alpha)\):

\[
d u = u'(\alpha) \, d \alpha
\]

Introduction of higher derivatives and differentials:

\[
d^2 u = d(d u) = u''(\alpha) \, d \alpha^2 = \frac{d^2 u(\alpha)}{d \alpha^2} \, d \alpha^2 \quad \text{etc.}
\]
(b) **VECTOR-VALUED FUNCTIONS OF SEVERAL VARIABLES**

It exists:

\[ u = u(\alpha, \beta, \gamma, ...) \quad \text{with} \quad \{\alpha, \beta, \gamma, ...\} : \text{real scalar variable} \]

Partial derivative of \( u(\alpha, \beta, \gamma, ...) \):

\[ w_\alpha(\alpha, \beta, \gamma, ...) := \frac{\partial u(\cdot)}{\partial \alpha} =: u_\alpha \]

Total differential of \( u(\alpha, \beta, \gamma, ...) \):

\[ du = u_{,\alpha} \, d\alpha + u_{,\beta} \, d\beta + u_{,\gamma} \, d\gamma + \cdots \]

Higher partial derivative (examples):

\[ u_{,\alpha\alpha} = \frac{\partial^2 u(\cdot)}{\partial \alpha^2} ; \quad u_{,\gamma\beta} = \frac{\partial^2 u(\cdot)}{\partial \gamma \partial \beta} \]

**Rem.:** The order of partial derivatives is permutable.

(c) **TENSOR FUNCTIONS OF A SINGLE OR OF SEVERAL VARIABLES**

**Rem.:** Tensor-valued functions are treated analogously to the above procedure.

(d) **DERIVATIVE OF PRODUCTS OF FUNCTIONS**

Some rules:

\[ (a \otimes b)' = a' \otimes b + a \otimes b' \]
\[ (A B)' = A' B + A B' \]
\[ (A^{-1})' = -A^{-1} A' A^{-1} \]

3.3 **Functions of vector and tensor variables**

(a) **THE GRADIENT OPERATOR**

**Rem.:** Functions of the position (placement) vector are called **field functions**. Derivatives with respect to the position vector are called “gradient of a function”.

Scalar-valued functions \( \phi(x) \)

\[ \text{grad} \, \phi(x) := \frac{d\phi(x)}{dx} =: w(x) \quad \rightarrow \quad \text{result is a vector field} \]

or in basis notation

\[ \text{grad} \, \phi(x) := \frac{\partial \phi(x)}{\partial x_i} e_i =: \phi_i \, e_i \]
Vector-valued functions $v(x)$

$$\text{grad } v(x) := \frac{d v(x)}{d x} =: S(x) \quad \rightarrow \quad \text{result is a tensor field}$$

or in basis notation

$$\text{grad } v(x) := \frac{\partial v_i(x)}{\partial x_j} e_i \otimes e_j =: v_{i,j} e_i \otimes e_j$$

Tensor-valued functions $T(x)$

$$\text{grad } T(x) := \frac{d T(x)}{d x} =: U(x) \quad \rightarrow \quad \text{result is a tensor field of 3-rd order}$$

or in basis notation

$$\text{grad } T(x) := \frac{\partial t_{ik}(x)}{\partial x_j} e_i \otimes e_k \otimes e_j =: t_{ik,j} e_i \otimes e_k \otimes e_j$$

Rem.: The gradient operator $\text{grad } (\cdot) = \nabla (\cdot)$ (with $\nabla$ : Nabla operator) increases the order of the respective function by one.

(b) Derivative of functions of arbitrary vectorial and tensorial variables

Rem.: Derivatives concerning the respective variables are built analogously to the preceding procedures, e.g.

$$\frac{\partial R(T, v)}{\partial T} = \frac{\partial R_{ij}(T, v)}{\partial t_{st}} e_i \otimes e_j \otimes e_s \otimes e_t$$

Some specific rules for the derivative of tensor functions with respect to tensors

For arbitrary 2-nd order tensors $A, B, C$, the following rules hold:

$$\frac{\partial (AB)}{\partial B} = (A \otimes I)^T$$
$$\frac{\partial (AB)}{\partial A} = (I \otimes B^T)^T$$
$$\frac{\partial (AA)}{\partial A} = (A \otimes I)^T + (I \otimes A)^T$$
$$\frac{\partial (A^T A)}{\partial A} = (A^T \otimes I)^T + (I \otimes A)^T$$
$$\frac{\partial (AA^T)}{\partial A} = (A \otimes I)^T + (I \otimes A)^T$$
$$\frac{\partial (A^T A^T)}{\partial A} = (I \otimes A^T)^T + (A^T \otimes I)^T$$
\[ \frac{\partial (ABC)}{\partial B} = (A \otimes C^T)^{23} \]
\[ \frac{\partial A^T}{\partial A} = (I \otimes I)^{24} \]
\[ \frac{\partial A^{-1}}{\partial A} = -(A^{-1} \otimes A^{-1})^{23} \]
\[ \frac{\partial A^{-1}}{\partial A} = -(A^{-1} \otimes A^{-1})^{24} \]
\[ \frac{\partial \hat{A}}{\partial A} = \det A [(A^{-1} \otimes A^{-1}) - (A^{-1} \otimes A^{-1})^T] \]
\[ \frac{\partial (\alpha \beta)}{\partial C} = \alpha \frac{\partial \beta}{\partial C} + \beta \frac{\partial \alpha}{\partial C} \]
\[ \frac{\partial (\alpha v)}{\partial C} = v \otimes \frac{\partial \alpha}{\partial C} + \alpha \frac{\partial v}{\partial C} \]
\[ \frac{\partial (\alpha A)}{\partial C} = A \otimes \frac{\partial \alpha}{\partial C} + \alpha \frac{\partial A}{\partial C} \]
\[ \frac{\partial (A v)}{\partial C} = \left[ \left( \frac{\partial A}{\partial C} \right)^{24} v \right]^T v + \left[ A \frac{\partial v}{\partial C} \right]^2 \]
\[ \frac{\partial (u \cdot v)}{\partial C} = \left[ \left( \frac{\partial u}{\partial C} \right)^{23} v \right]^T + \left[ \left( \frac{\partial v}{\partial C} \right)^{23} u \right]^T \]
\[ \frac{\partial (A \cdot B)}{\partial C} = \left( \frac{\partial A}{\partial C} \right)^T B + \left( \frac{\partial B}{\partial C} \right)^T A \]
\[ \frac{\partial (AB)}{\partial C} = \left( \left( \frac{\partial A}{\partial C} \right)^{24} B \right)^{24} + \left( \left( \frac{\partial B}{\partial C} \right)^{24} A^T \right)^{24} \]

Furthermore,
\[ \frac{\partial A}{\partial A} = (I \otimes I)^{23} : I \]
\[ \frac{\partial A^T}{\partial A} = (I \otimes I)^{24} \]
\[ \frac{\partial (A \cdot I)}{\partial A} = (I \otimes I) \]
\[ \frac{\partial t(A)}{\partial A} = -\frac{1}{2} E \]

**Principal invariants and their derivatives** (see also section 2.9)
\[ \frac{\partial I_A}{\partial A} = I \quad \text{with} \quad I_A = A \cdot I \]
\[ \frac{\partial II_A}{\partial A} = A \otimes I \quad \text{with} \quad II_A = \frac{1}{2} (I_A^2 - A \cdot I) \]
\[ \frac{\partial III_A}{\partial A} = \hat{A} \quad \text{with} \quad III_A = \det A \]
Specific operators

Introduction of the further differential operators \( \text{div}(\cdot) \) and \( \text{rot}(\cdot) \).

**Divergence of a vector field** \( \mathbf{v}(\mathbf{x}) \)

\[
\text{div} \mathbf{v}(\mathbf{x}) := \text{grad} \mathbf{v}(\mathbf{x}) \cdot \mathbf{I} =: \phi(\mathbf{x}) \quad \rightarrow \quad \text{result is a scalar field}
\]

or in basis notation

\[
\text{div} \mathbf{v}(\mathbf{x}) = v_{i,j} \ (e_i \otimes e_j) \cdot (e_n \otimes e_n) = v_{n,n} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}
\]

**Divergence of a tensor field** \( \mathbf{T}(\mathbf{x}) \)

\[
\text{div} \mathbf{T}(\mathbf{x}) = [\text{grad} \mathbf{T}(\mathbf{x})] \mathbf{I} =: \mathbf{v}(\mathbf{x}) \quad \rightarrow \quad \text{result is a vector field}
\]

or in basis notation

\[
\text{div} \mathbf{T}(\mathbf{x}) = t_{ik,j} \ (e_i \otimes e_k \otimes e_j) (e_n \otimes e_n) = t_{in,m} e_i
\]

**Rem.:** The divergence operator \( \text{div}(\cdot) = \nabla \cdot (\cdot) \) decreases the order of the respective function by one.

**Rotation of a vector field** \( \mathbf{v}(\mathbf{x}) \)

\[
\text{rot} \mathbf{v}(\mathbf{x}) := \mathbf{E} [\text{grad} \mathbf{v}(\mathbf{x})]^T =: \mathbf{r}(\mathbf{x}) \quad \rightarrow \quad \text{result is a vector field}
\]

or in basis notation

\[
\text{rot} \mathbf{v}(\mathbf{x}) = e_{ijn} \ (e_i \otimes e_j \otimes e_n) v_{o,p} \ (e_p \otimes e_o) = e_{ijn} v_{n,j} e_i
\]

**Consequence:** \( \text{rot} \mathbf{v}(\mathbf{x}) \) yields twice the axial vector corresponding to the skew-symmetric part of \( \text{grad} \mathbf{v}(\mathbf{x}) \).

**Rem.:** The rotation operator \( \text{rot}(\cdot) = \text{curl}(\cdot) = \nabla \times (\cdot) \) preserves the order of the respective function.

**Laplace operator**

\[
\Delta(\cdot) := \text{div} \text{grad} (\cdot) \quad \rightarrow \quad \text{analogical to the precedings}
\]

**Rem.:** The Laplace operator \( \Delta(\cdot) = \nabla \cdot \nabla (\cdot) \) preserves the order of the differentiated function.

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Rules for the operators $\text{grad} \left( \cdot \right)$, $\text{div} \left( \cdot \right)$, and $\text{rot} \left( \cdot \right)$

\[
\begin{align*}
\text{grad} \left( \phi \psi \right) &= \phi \text{grad} \psi + \psi \text{grad} \phi \\
\text{grad} \left( \phi \mathbf{v} \right) &= \mathbf{v} \otimes \text{grad} \phi + \phi \text{grad} \mathbf{v} \\
\text{grad} \left( \phi \mathbf{T} \right) &= \mathbf{T} \otimes \text{grad} \phi + \phi \text{grad} \mathbf{T} \\
\text{grad} \left( \mathbf{u} \cdot \mathbf{v} \right) &= (\text{grad} \mathbf{u})^T \mathbf{v} + (\text{grad} \mathbf{v})^T \mathbf{u} \\
\text{grad} \left( \mathbf{u} \times \mathbf{v} \right) &= \mathbf{u} \times \text{grad} \mathbf{v} + \text{grad} \mathbf{u} \times \mathbf{v} \\
\text{grad} \left( \mathbf{a} \otimes \mathbf{b} \right) &= [\text{grad} \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes (\text{grad} \mathbf{b})^T]^T \\
\text{grad} \left( \mathbf{T} \mathbf{v} \right) &= (\text{grad} \mathbf{T})^T \mathbf{v} + \mathbf{T} \text{grad} \mathbf{v} \\
\text{grad} \left( \mathbf{T} \mathbf{S} \right) &= [(\text{grad} \mathbf{T})^T \mathbf{S}]^T + (\mathbf{T} \text{grad} \mathbf{S})^T \\
\text{grad} \left( \mathbf{T} \cdot \mathbf{S} \right) &= (\text{grad} \mathbf{T})^T \mathbf{S}^T + (\text{grad} \mathbf{S})^T \mathbf{T}^T \\
\text{grad} \mathbf{x} &= \mathbf{I} \\
\text{div} \left( \mathbf{u} \otimes \mathbf{v} \right) &= \mathbf{u} \text{div} \mathbf{v} + (\text{grad} \mathbf{u}) \mathbf{v} \\
\text{div} \left( \phi \mathbf{v} \right) &= \mathbf{v} \cdot \text{grad} \phi + \phi \text{div} \mathbf{v} \\
\text{div} \left( \mathbf{T} \mathbf{v} \right) &= (\text{div} \mathbf{T}^T) \cdot \mathbf{v} + \mathbf{T}^T \cdot \text{grad} \mathbf{v} \\
\text{div} \left( \text{grad} \mathbf{v} \right)^T &= \text{grad} \text{div} \mathbf{v} \\
\text{div} \left( \mathbf{u} \times \mathbf{v} \right) &= (\text{grad} \mathbf{u} \times \mathbf{v}) \cdot \mathbf{I} - (\text{grad} \mathbf{v} \times \mathbf{u}) \cdot \mathbf{I} \\
&= \mathbf{v} \cdot \text{rot} \mathbf{u} - \mathbf{u} \cdot \text{rot} \mathbf{v} \\
\text{div} \left( \phi \mathbf{T} \right) &= \mathbf{T} \text{grad} \phi + \phi \text{div} \mathbf{T} \\
\text{div} \left( \mathbf{T} \mathbf{S} \right) &= (\text{grad} \mathbf{T}) \mathbf{S} + \mathbf{T} \text{div} \mathbf{S} \\
\text{div} \left( \mathbf{v} \times \mathbf{T} \right) &= \mathbf{v} \times \text{div} \mathbf{T} + \text{grad} \mathbf{v} \times \mathbf{T} \\
\text{div} \left( \mathbf{v} \otimes \mathbf{T} \right) &= \mathbf{v} \otimes \text{div} \mathbf{T} + (\text{grad} \mathbf{v}) \mathbf{T}^T \\
\text{div} \left( \mathbf{v} \otimes \mathbf{T} \right)^T &= \mathbf{v} \otimes \text{div} \mathbf{T}^T + [(\text{grad} \mathbf{v}) (\mathbf{T}^T)^T]^T \\
\text{div} \left( \text{grad} \mathbf{v} \right)^+ &= 0 \\
\text{div} \left[ \text{grad} \mathbf{v} \pm (\text{grad} \mathbf{v})^T \right] &= \text{div} \text{grad} \mathbf{v} \pm \text{grad} \text{div} \mathbf{v} \\
\text{div} \text{rot} \mathbf{v} &= 0 \\
\text{rot} \text{rot} \mathbf{v} &= \text{grad} \text{div} \mathbf{v} - \text{div} \text{grad} \mathbf{v} \\
\text{rot} \text{grad} \phi &= 0
\end{align*}
\]
\[
\text{rot } \text{grad } \mathbf{v} = 0 \\
\text{rot } (\text{grad } \mathbf{v})^T = \text{grad } \text{rot } \mathbf{v} \\
\text{rot } (\phi \mathbf{v}) = \phi \text{rot } \mathbf{v} + \text{grad } \phi \times \mathbf{v} \\
\text{rot } (\mathbf{u} \times \mathbf{v}) = \text{div } (\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}) = \mathbf{u} \text{div } \mathbf{v} + (\text{grad } \mathbf{u}) \times \mathbf{v} - \text{div } \mathbf{u} - (\text{grad } \mathbf{v}) \mathbf{u}
\]

**Grassmann evolution:**

\[
\mathbf{v} \times \text{rot } \mathbf{v} = \frac{1}{2} \text{grad } (\mathbf{v} \cdot \mathbf{v}) - (\text{grad } \mathbf{v}) \mathbf{v} = (\text{grad } \mathbf{v})^T \mathbf{v} - (\text{grad } \mathbf{v}) \mathbf{v}
\]

### 3.4 Integral theorems

**Rem.:** In what follows, some integral theorems for the transformation of surface integrals into volume integrals are presented.

**Requirement:** \( \mathbf{u} = \mathbf{u}(\mathbf{x}) \) is a steady and sufficiently often steadily differentiable vector field. The domain of \( \mathbf{u} \) is in \( \mathcal{V}^3 \).

**Proof of the integral theorem**

\[
\int_S \mathbf{u}(\mathbf{x}) \otimes d\mathbf{a} = \int_V \text{grad } \mathbf{u}(\mathbf{x}) d\mathbf{v} \quad \text{with } d\mathbf{a} = n \, da
\]

and \[ \{ \begin{array}{l} \text{da} : \text{surface element} \\ \mathbf{n} : \text{outward oriented unit surface normal vector} \end{array} \]

**Basis:** Consideration of an infinitesimal volume element \( d\mathbf{v} \) spanned in the point \( X \) by the position vector \( \mathbf{x} \), and \( \bar{\mathbf{u}}_i \), i.e. the values of \( \mathbf{u}(\mathbf{x}) \) in the centroid of the partial surfaces 1 - 6.
Determination of the surface element vectors \( da_i \):
\[
da_1 = dx_2 \times dx_3 = dx_2 \, dx_3 \, (e_2 \times e_3) = dx_2 \, dx_3 \, e_1 = -da_4 \rightarrow e_1 = n_1 = -n_4
\]
Furthermore, one obtains
\[
da_2 = dx_3 \, dx_1 \, e_2 = -da_5 \rightarrow e_2 = n_2 = -n_5
\]
\[
da_3 = dx_1 \, dx_2 \, e_3 = -da_6 \rightarrow e_3 = n_3 = -n_6
\]

**Rem.:** The surface vectors hold the condition \( \sum_{i=1}^{6} da_i = 0 \).

Determination of the volume elements \( dv \):
\[
dv = (dx_1 \times dx_2) \cdot dx_3 = dx_1 \, dx_2 \, dx_3
\]

Values of \( u(x) \) in the centroids of the partial surfaces:

**Rem.:** The increments of \( u(x) \) in the directions of \( dx_1, dx_2, dx_3 \) are approximated by the first term of a Taylor series.
\[
\bar{u}_1 = u(x) + \frac{1}{2} \frac{\partial u}{\partial x_2} \, dx_2 + \frac{1}{2} \frac{\partial u}{\partial x_3} \, dx_3
\]
\[
\bar{u}_1 = u_4 + \frac{\partial u}{\partial x_1} \, dx_1
\]
Furthermore, one obtains
\[
\bar{u}_2 = u_5 + \frac{\partial u}{\partial x_2} \, dx_2 , \quad \bar{u}_3 = u_6 + \frac{\partial u}{\partial x_3} \, dx_3
\]

Computation of the surface integral yields
\[
\int_{S(dv)} u(x) \otimes da \rightarrow \sum_{i=1}^{6} \bar{u}_i \otimes da_i = \bar{u}_1 \otimes da_1 + \bar{u}_4 \otimes da_4 + \cdots
\]
\[
(u_1 \frac{\partial u}{\partial x_1} \, dx_1) \otimes (-da_1)
\]
Thus
\[
\sum_{i=1}^{6} \bar{u}_i \otimes da_i = \frac{\partial u}{\partial x_1} \, dx_1 \otimes da_1 + \frac{\partial u}{\partial x_2} \, dx_2 \otimes da_2 + \frac{\partial u}{\partial x_3} \, dx_3 \otimes da_3
\]
with
\[
da_1 = dx_2 \, dx_3 \, e_1 , \quad da_2 = dx_1 \, dx_3 \, e_2 , \quad da_3 = dx_1 \, dx_2 \, e_3
\]
yields
\[
\sum_{i=1}^{6} \bar{u}_i \otimes da_i = \left( \frac{\partial u}{\partial x_1} \otimes e_1 + \frac{\partial u}{\partial x_2} \otimes e_2 + \frac{\partial u}{\partial x_3} \otimes e_3 \right) \frac{dx_1 \, dx_2 \, dx_3}{dv} \quad \text{grad } u
\]
Thus
\[ \sum_{i=1}^{6} \mathbf{u}_i \otimes d\mathbf{a}_i = \text{grad } \mathbf{u} \, dv \]

Integration over an arbitrary volume \( V \) yields
\[ \int_S \mathbf{u}(\mathbf{x}) \otimes d\mathbf{a} = \int_V \text{grad } \mathbf{u}(\mathbf{x}) \, dv \quad \text{q. e. d.} \quad (*) \]

(b) **PROOF OF THE GAUSSIAN INTEGRAL THEOREM**

\[ \int_S \mathbf{u}(\mathbf{x}) \cdot d\mathbf{a} = \int_V \text{div } \mathbf{u}(\mathbf{x}) \, dv \]

**Basis:** Integral theorem (*) after scalar multiplication with the identity tensor

\[ I \cdot \int_S \mathbf{u}(\mathbf{x}) \otimes d\mathbf{a} = \int_V I \cdot \text{grad } \mathbf{u}(\mathbf{x}) \, dv \]

\[ \rightarrow \int_S I \cdot [\mathbf{u}(\mathbf{x}) \otimes d\mathbf{a}] = \int_V I \cdot \text{grad } \mathbf{u}(\mathbf{x}) \, dv \]

Thus, leading to
\[ \int_S \mathbf{u}(\mathbf{x}) \cdot d\mathbf{a} = \int_V \text{div } \mathbf{u}(\mathbf{x}) \, dv \quad (**) \]

(c) **PROOF OF THE INTEGRAL THEOREM**

\[ \int_S \mathbf{T}(\mathbf{x}) \, d\mathbf{a} = \int_V \text{div } \mathbf{T}(\mathbf{x}) \, dv \]

**Basis:** Scalar multiplication of the surface integral with a constant vector \( \mathbf{b} \in \mathbb{R}^3 \)

\[ \mathbf{b} \cdot \int_S \mathbf{T}(\mathbf{x}) \, d\mathbf{a} = \int_S \mathbf{b} \cdot \mathbf{T}(\mathbf{x}) \, d\mathbf{a} = \int_S [\mathbf{T}^T(\mathbf{x}) \mathbf{b}] \cdot d\mathbf{a} =: \int_S \mathbf{u}(\mathbf{x}) \cdot d\mathbf{a} \]

with \( \mathbf{u}(\mathbf{x}) := \mathbf{T}^T(\mathbf{x}) \mathbf{b} \)

It follows with the integral theorem (**) \[ \mathbf{b} \cdot \int_S \mathbf{T}(\mathbf{x}) \, d\mathbf{a} = \int_V \text{div } [\mathbf{T}^T(\mathbf{x}) \mathbf{b}] \, dv \]

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In particular, with $b = \text{const.}$ and a divergence rule follows

$$\text{div} \left[T^T(x) b\right] = \text{div} T(x) \cdot b$$

leading to

$$b \cdot \int_S T(x) \, da = \int_V \text{div} T(x) \cdot b \, dv$$

Thus

$$\int_S T(x) \, da = \int_V \text{div} T(x) \, dv \quad \text{q. e. d.}$$

**Rem.:** At this point, no further proofs are carried out.

**Summary of some integral theorems**

For the transformation of surface integrals into volume integrals, the following relations hold:

$$\int_S u \otimes da = \int_V \text{grad} u \, dv$$

$$\int_S \phi \, da = \int_V \text{grad} \phi \, dv$$

$$\int_S u \cdot da = \int_V \text{div} u \, dv$$

$$\int_S u \times da = - \int_V \text{rot} u \, dv$$

$$\int_S T \, da = \int_V \text{div} T \, dv$$

$$\int_S u \times T \, da = \int_V \text{div} (u \times T) \, dv$$

$$\int_S u \otimes T \, da = \int_V \text{div} (u \otimes T) \, dv$$

For the transformation of line into surface integrals the following relations hold:
\[ \oint_L u \otimes dx = - \int_S \text{grad} u \times da \]
\[ \oint_L \phi dx = - \int_S \text{grad} \phi \times da \]
\[ \oint_L u \cdot dx = \int_S (\text{rot} u) \cdot da \]
\[ \oint_L u \times dx = \int_S (I \text{div} u - \text{grad}^T u) \, da \]
\[ \oint_L T dx = \int_S (\text{rot} T)^T \, da \]

with \( da = n \, da \)

**Rem.**: If required, further relations of the vector and tensor calculus will be presented in the respective context. The description of non-orthogonal and non-unit basis systems was not discussed in this contribution.
3.5 Transformations between actual and reference configurations

Given are the deformation gradient \( F = \partial \mathbf{x} / \partial \mathbf{X} \) and arbitrary vectorial and tensorial field functions \( \mathbf{v} \) and \( \mathbf{A} \). Then, with

\[
\begin{align*}
\text{reference configuration} & : \\
\text{Grad} \ (\cdot) & = \frac{\partial}{\partial \mathbf{X}} (\cdot) \\
\text{Div} \ (\cdot) & = [\text{Grad} \ (\cdot)] \cdot \mathbf{I} \quad \text{or} \quad [\text{Grad} \ (\cdot)] \mathbf{I} \\
\text{actual configuration} & : \\
\text{grad} \ (\cdot) & = \frac{\partial}{\partial \mathbf{x}} (\cdot) \\
\text{div} \ (\cdot) & = [\text{grad} \ (\cdot)] \cdot \mathbf{I} \quad \text{or} \quad [\text{grad} \ (\cdot)] \mathbf{I}
\end{align*}
\]

the following relations hold:

\[
\begin{align*}
\text{Grad} \ \mathbf{v} & = (\text{grad} \ \mathbf{v}) \ F \\
\text{grad} \ \mathbf{v} & = (\text{Grad} \ \mathbf{v}) \ F^{-1} \\
\text{Div} \ \mathbf{v} & = (\text{grad} \ \mathbf{v}) \cdot F^T \\
\text{div} \ \mathbf{v} & = (\text{Grad} \ \mathbf{v}) \cdot F^{T-1}
\end{align*}
\]

\[
\begin{align*}
\text{Grad} \ \mathbf{A} & = [(\text{grad} \ \mathbf{A}) \ F]_2 \\
\text{grad} \ \mathbf{A} & = [(\text{Grad} \ \mathbf{A}) \ F^{-1}]_2 \\
\text{Div} \ \mathbf{A} & = (\text{grad} \ \mathbf{A}) \ F^T \\
\text{div} \ \mathbf{A} & = (\text{Grad} \ \mathbf{A}) \ F^{T-1}
\end{align*}
\]

Furthermore, it can be shown that

\[
\begin{align*}
\text{Div} \ F^{T-1} & = -F^{T-1} (F^{T-1} \ \text{Grad} \ F)_2 = -(\det F)^{-1} F^{T-1} [\text{Grad} \ (\det F)] \\
\text{div} \ F^T & = -F^T (F^T \ \text{grad} \ F^{-1})_2 = -(\det F) F^T [\text{grad} \ (\det F)^{-1}]
\end{align*}
\]